Sums of reciprocals modulo composite integers

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Joint work with

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1. Introduction

Since $\{1, 2, \ldots, p-1\}$ forms a reduced residue system mod p (an odd prime), so does $\{1, 1/2, \ldots, 1/(p-1)\}$, and therefore we have

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What can be said about *partial* sums?
Eisenstein (1850) showed

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j} \equiv -2\, q_p(2) \pmod{p},$$

where $q_p(a)$ is the *Fermat quotient* to base a ($p \nmid a$), defined for odd primes p by

$$q_p(a) := \frac{a^{p-1} - 1}{p}.$$

This was later extended in various directions, among them:

(1) Modulo higher powers of p, e.g., (Emma Lehmer, 1938)

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j} \equiv -2\, q_p(2) + p\, q_p(2)^2 \pmod{p^2}.$$

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Typically there exist explicit expressions for such congruences for sums of lenght $\lfloor \frac{p}{2} \rfloor$, $\lfloor \frac{p}{3} \rfloor$, $\lfloor \frac{p}{4} \rfloor$, and $\lfloor \frac{p}{6} \rfloor$.

Reason: Bernoulli polynomials are usually involved.

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(2) More recently: A mod p^3 extension of a theorem of Gauss:

Let p and a be such that $p \equiv 1 \pmod 4$, $p = a^2 + b^2$, $a \equiv 1 \pmod 4$. Then

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p.$$

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and further by John Cosgrave and KD (2010):

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(2a - \frac{p}{2a} - \frac{p^2}{8a^3}\right) \times \left(1 + \tfrac{1}{2}pq_p(2) + \tfrac{1}{8}p^2\left(2E_{p-3} - q_p(2)^2\right)\right) \pmod{p^3}.$$

Here E_n denotes the nth Euler number (see below).

In the proof of this last extension, numerous congruences of “Lehmer type" were needed.

The congruence

$$\sum_{j=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{j} \equiv -3\, q_p(2) \pmod{p}$$

is a special case of a sum over an arithmetic progression:

$$\sum_{j=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{p-4j} \equiv \frac{3}{4} q_p(2) - \frac{3}{8} p\, q_p(2)^2 \pmod{p^2}$$

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First goal of this talk:

Study extensions of these to composite moduli.

2. Composite Moduli

Congruences modulo *composite* integers first obtained independently by H. F. Baker and M. Lerch (1906).

However, it appears that the first composite analogue of a “Lehmer type" congruence was only published in 2002 (T. Cai):

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However, it appears that the first composite analogue of a “Lehmer type" congruence was only published in 2002 (T. Cai): For any odd $n > 1$,

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\frac{n-1}{2}} \frac{1}{j} \equiv -2\, q_n(2) + n\, q_n(2)^2 \pmod{n^2},$$

where $q_n(a)$ is the *Euler quotient* of n with base a defined by

$$q_n(a) := \frac{a^{\varphi(n)} - 1}{n} \qquad (\gcd(a, n) = 1, n > 1).$$

3. Interlude: Euler quotients

First introduced and studied by Lerch (1905).

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(3) $q_{mn}(a) \equiv \frac{\varphi(n)}{n} q_m a \pmod{m}$.

(with reasonable restrictions on a, b, m, n).

Another interesting property:

Theorem 1 (Baker, Lerch)

Let $a \geq 1$, $n \geq 2$ with $\gcd(a, n) = 1$. Then

$$q_n(a) \equiv \sum_{\substack{r=1 \\ \gcd(r,n)=1}}^{n-1} \frac{\lambda(r)}{r} \pmod{n},$$

where $\lambda(r)$ is the least nonnegative residue of $-r/n \pmod{a}$.

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Independently published by Baker and Lerch in 1906;
Special case (a, n prime) due to Sylvester (1861).

Connections with Bernoulli numbers and polynomials:

Recall: Bernoulli numbers can be defined by

$$\frac{t}{e^t - 1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} t^r.$$

Then $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_{2k+1} = 0$ for $k \geq 1$.

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Key property: For integers $k \geq 1$,

$$\sum_{j=1}^{n-1} j^k = \frac{1}{k+1} \left(B_{k+1}(n) - B_{k+1} \right).$$

The following establishes the link between sums of powers (including sums of reciprocals – by Fermat's Little theorem) and Euler quotients:

Lemma 2

Let $a \geq 1$*,* $n \geq 2$ *with* $\gcd(a,n) = 1$*. Then*

$$aq_n(a) = -\frac{a^{\varphi(m)}}{mB_{\varphi(m)}} \sum_{j=1}^{a-1} \left(B_{\varphi(m)}(\frac{j}{a}) - B_{\varphi(m)} \right).$$

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Proof: Well-known identities for Bernoulli polynomials.

Lemma extends similar result for prime n.

4. Composite Moduli (Cont'd)

The following extensions of 4 congruences of Emma Lehmer were recently obtained by Cai, Fu and Zhou (2007) and independently by Cao and Pan (2009):

For any odd $n \geq 1$ with $n \not\equiv 0 \pmod 3$ we have

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n-2j} \equiv q_n(2) - \frac{1}{2} n q_n(2)^2 \pmod{n^2}, \tag{1}$$

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\lfloor \frac{n}{3} \rfloor} \frac{1}{n-3j} \equiv \frac{1}{2} q_n(3) - \frac{1}{4} n\, q_n(3)^2 \pmod{n^2}, \tag{2}$$

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{n-4j} \equiv \frac{3}{4} q_n(2) - \frac{3}{8} n\, q_n(2)^2 \pmod{n^2}, \tag{3}$$

and

$$\sum_{\substack{j=1\\(j,n)=1}}^{\lfloor\frac{n}{6}\rfloor} \frac{1}{n-6j} \equiv \frac{1}{3}q_n(2)+\frac{1}{4}q_n(3) - n\left(\frac{1}{6}q_n(2)^2+\frac{1}{8}q_n(3)^2\right) \pmod{n^2}. \qquad (4)$$

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Note: The full restriction "n odd and $n\not\equiv 0 \pmod 3$" is only needed in (4)

(1) and (3) make sense also when $n\equiv 0 \pmod 3$, and (2) makes sense for *even n*.

and

$$\sum_{\substack{j=1\\(j,n)=1}}^{\lfloor \frac{n}{6} \rfloor} \frac{1}{n-6j} \equiv \frac{1}{3}q_n(2) + \frac{1}{4}q_n(3) - n\left(\frac{1}{6}q_n(2)^2 + \frac{1}{8}q_n(3)^2\right) \pmod{n^2}. \tag{4}$$

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Questions:
(a) For which n are (1)–(3) correct after all?

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Questions:
(a) For which n are (1)–(3) correct after all?
(b) Otherwise, what are the correct statements for (1)–(3)?

5. Congruences (1) and (3) for $3 \mid n$

Cao and Pan (2009) wrote that (3) "fails when $n = 9, 15, 27, 33, 45, 51, 69, 75, 81, 87, \ldots$".

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Theorem 3

For odd $n \geq 1$ with $3 \mid n$, the congruences

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n-2j} \equiv q_n(2) - \frac{1}{2} n q_n(2)^2 \pmod{n^2},$$

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If n has no such prime divisor, then they hold modulo $\frac{1}{3}n^2$.

Idea of proof:

- Inclusion/exclusion principle.
- Congruences for Euler quotients.
- A deep property of Bernoulli numbers (see below).
- The fact that $3 \mid \varphi(m)$ iff m has a prime divisor $p \equiv 1 \pmod{6}$.
- The Chinese Remainder Theorem.

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Lemma 4 (Carlitz, 1953, 1960)

(1) For any prime p and $\beta \geq 1$ we have

$$pB_{\varphi(p^\beta)} \equiv p - 1 \pmod{p^\beta},$$

with the exception of the pair $p = 2$, $\beta = 2$, where we have $2B_2 \equiv -1 \pmod{2^2}$.

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with the exception of the pair $p = 2$, $\beta = 2$, where we have $2B_2 \equiv -1 \pmod{2^2}$.

(2) Let p be a prime and $n > 1$ such that $(p-1)p^h \mid 2n$. Then

$$pB_{2n} \equiv p - 1 \pmod{p^{h+1}}.$$

6. The Congruence (2) for even n

Cao and Pan (2009) wrote that (2) "fails when $n = 4, 8, 14, 16, 22, 28, 32, 38, 44, 46, \ldots$"

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Theorem 5

For positive integers $n \equiv \pm 2 \pmod 6$ the congruence

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holds iff n has a prime divisor $p \equiv 1 \pmod 4$ or two distinct odd prime divisors.

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(a) *modulo $\frac{1}{2}n^2$ when $n = 2^\alpha q^\beta$ for a prime $q \equiv 3 \pmod 4$ and $\alpha, \beta \geq 1$;*

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Outline of proof is similar to that of previous theorem.

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Main criterion comes from the fact that $\varphi(m) \equiv 0 \pmod 4$ if
- m has a prime factor $p \equiv 1 \pmod 4$, or
- m has at least two distinct odd prime factors.

A consequence: Different type of E. Lehmer's congruences:

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv 0 \pmod{p},$$

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Composite analogue:

Corollary 6

For any odd integer $n \geq 3$ we have

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\frac{n-1}{2}} \frac{1}{j^2} \equiv 0 \pmod{n},$$

unless $3 \mid n$ and n has no prime divisor $p \equiv 1 \pmod{6}$, in which case congruence holds modulo $n/3$.

7. Another Lehmer congruence

Emma Lehmer also proved the following: for primes $p \geq 5$,

$$\sum_{j=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{j^2} \equiv (-1)^{\frac{p-1}{2}} 4E_{p-3} \pmod{p},$$

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Euler numbers are integers, and the first few are

$E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, and $E_{2j+1} = 0$ for $j \geq 0$.

Lehmer's congruence

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was extended to prime powers by Cai, Fu and Zhou (2007): for odd primes p and integers $\alpha \geq 1$,

$$\sum_{\substack{j=1 \\ p \nmid j}}^{\lfloor p^\alpha/4 \rfloor} \frac{1}{j^2} \equiv (-1)^{\frac{p^\alpha-1}{2}} 4E_{\varphi(p^\alpha)-2} \begin{cases} \pmod{p^\alpha} & \text{when} \quad p \geq 5, \\ \pmod{3^{\alpha-1}} & \text{when} \quad p = 3. \end{cases}$$

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There's no obvious extension to arbitrary odd moduli.

Lehmer's congruence

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was extended to prime powers by Cai, Fu and Zhou (2007): for odd primes p and integers $\alpha \geq 1$,

$$\sum_{\substack{j=1 \\ p\nmid j}}^{\lfloor p^\alpha/4 \rfloor} \frac{1}{j^2} \equiv (-1)^{\frac{p^\alpha-1}{2}} 4E_{\varphi(p^\alpha)-2} \begin{cases} \pmod{p^\alpha} & \text{when} \quad p \geq 5, \\ \pmod{3^{\alpha-1}} & \text{when} \quad p = 3. \end{cases}$$

There's no obvious extension to arbitrary odd moduli.

Goal of this part of the talk: To find such an extension.

8. Interlude: Euler numbers

Recall: Euler numbers are defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.$$

Odd-index Euler numbers are 0; first few even-index ones are $1, -1, 5, -61, 1385, -50521$.

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Numerous generalizations and extensions are known.

We'll extend this to arbitrary odd moduli.

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This generalizes the concept of a square-free integer.

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Lemma 7

Let $k \geq 1$ *and* $n \geq 1$ *an odd* $(k+1)$*th-power free integer. Then*

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Method of proof: Use the congruence

$$E_m \equiv \sum_{j=0}^{n-1} (-1)^j (2j+1)^m \pmod{n},$$

valid for arbitrary integers $m \geq 1$ and odd integers $n \geq 1$. (Carlitz, 1954).

Then use the following extension of Euler's theorem:

Lemma 8

Let $n, k \in \mathbb{N}$. Then

$$a^{\varphi(n)+k} \equiv a^k \pmod{n} \qquad \textit{for all} \quad a \in \mathbb{Z}$$

iff n is a $(k+1)$th-power free integer.

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iff n *is a* $(k+1)$*th-power free integer.*

Proof is elementary and uses the Chinese Remainder Theorem again.

For the main result we need the following function of n. With

$$n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$$

define $A(n) \in \mathbb{N}$ by $A(n) = 1$ when $r = 1$ and for $r \geq 2$,

$$A(n) := \sum_{j=1}^{r} \prod_{\substack{i=1 \\ i \neq j}}^{r} p_i^{\alpha_i \varphi(p_j^{\alpha_j})} \left(1 - \frac{(-1)^{(p_i-1)/2}}{p_i^2} \right).$$

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Theorem 9

Let $n \in \mathbb{N}$ be odd. Then

$$\sum_{\substack{j=1 \\ (j,n)=1}}^{\lfloor n/4 \rfloor} \frac{1}{j^2} \equiv \begin{cases} (-1)^{\frac{n-1}{2}} 4A(n) E_{\varphi(n)-2} \pmod{n}, & 3 \nmid n, \\ (-1)^{\frac{n-1}{2}} 4A(n) E_{\varphi(n)-2} \pmod{n/3}, & n \equiv 0 \ (9), \\ (-1)^{\frac{n-1}{2}} \frac{40}{9} A(\frac{n}{3}) E_{\varphi(n)-2} \pmod{n/3}, & n \equiv \pm 3 \ (9). \end{cases}$$

Outline of proof:

- For each prime power $p^\alpha \mid n$, divide $\lfloor \frac{n}{4} \rfloor$ by p^α with remainder.
- Use inclusion/exclusion (via the Möbius function).
- Use the (known) congruence for prime powers.
- Use the extended Kummer congruence for Euler numbers.
- Combine everything with the Chinese Remainder Theorem.
- Particular care needs to be taken with powers of 3.

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$$\sum_{\substack{j=1\\(j,n)=1}}^{\lfloor n/4\rfloor} \frac{1}{j^2} \equiv 0 \pmod{n}?$$

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Looking at the theorem:

$$\sum_{\substack{j=1\\(j,n)=1}}^{\lfloor n/4\rfloor} \frac{1}{j^2} \equiv \begin{cases} (-1)^{\frac{n-1}{2}} 4A(n)E_{\varphi(n)-2} \pmod{n}, & 3 \nmid n, \\ (-1)^{\frac{n-1}{2}} 4A(n)E_{\varphi(n)-2} \pmod{n/3}, & n \equiv 0\ (9), \\ (-1)^{\frac{n-1}{2}} \frac{40}{9}A(\frac{n}{3})E_{\varphi(n)-2} \pmod{n/3}, & n \equiv \pm 3\ (9), \end{cases}$$

Can we have $A(n) \equiv 0 \pmod{n}$?

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If for odd $n \in \mathbb{N}$ *we have* $A(n) \equiv 0 \pmod{n}$ *then* $3 \mid n$ *but* $9 \nmid n$.

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For a proof, we need the following two lemmas.

Lemma 11

For an odd $n \in \mathbb{N}$ we have $A(n) \equiv 0 \pmod{n}$ iff

$$\prod_{\substack{i=1 \\ i \neq j}}^{r} \left(p_i^2 - (-1)^{(p_i-1)/2}\right) \equiv 0 \pmod{p_j^{\alpha_j}} \qquad \text{for all} \quad j = 1, \dots, r$$

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Sketch of proof:

- Consider congruences $\pmod{p_j^{\alpha_j}}$ separately.
- Euler's generalization of Fermat's Little Theorem.
- Count/estimate exponents of p_j.

Lemma 12

Suppose that $n \not\equiv \pm 3 \pmod 9$ *and* $A(n) \equiv 0 \pmod n$*. Then* n *has two prime factors* $p < q$ *with* $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 4$, *and*

$$p^2 + 1 \equiv 0 \pmod q,$$
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The following result shows that this is impossible.
It is of interest in its own right:

Theorem 13

For $\delta = \pm 1$ and $\varepsilon = \pm 1$, consider the pair of congruences

$$\begin{cases} p^2 & \equiv \delta \pmod{q}, \\ q^2 & \equiv \varepsilon \pmod{p}, \end{cases}$$

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(b) *If $\delta = -1, \varepsilon = 1$, then $(p, q) = (3, 5)$ is the only solution.*

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(c) *If $\delta = \varepsilon = -1$, then the only solutions are $(p, q) = (F_n, F_{n+2})$, $n = 1, 2, \ldots$, provided both Fibonacci number F_n, F_{n+2} are prime.*

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Part (b) is the case of Lemma 12.

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The only prime pairs F_n, F_{n+2} occur when

- $n = 5$: $(p, q) = (5, 13)$;
- $n = 11$: $(p, q) = (89, 233)$;
- $n = 431$: (p, q) have 90 and 91 decimal digits;
- $n = 569$: (p, q) both have 119 digits.

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If there are further solutions, then both primes will have more than 470 849 digits.

Outline of Proof of Theorem 13:

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- Use the fact that these sequences are "divisibility sequences", i.e.,

$$n \mid m \quad \Rightarrow \quad U_n(P,Q) \mid U_m(P,Q) \quad \text{and} \quad V_n(P,Q) \mid V_m(P,Q).$$

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- This shows that there are either

– no solutions, or
– just one solution (e.g., $(F_4, F_5) = (3, 5)$), or
– a class of solutions (e.g., pairs of Fibonacci primes).

Final Remark:

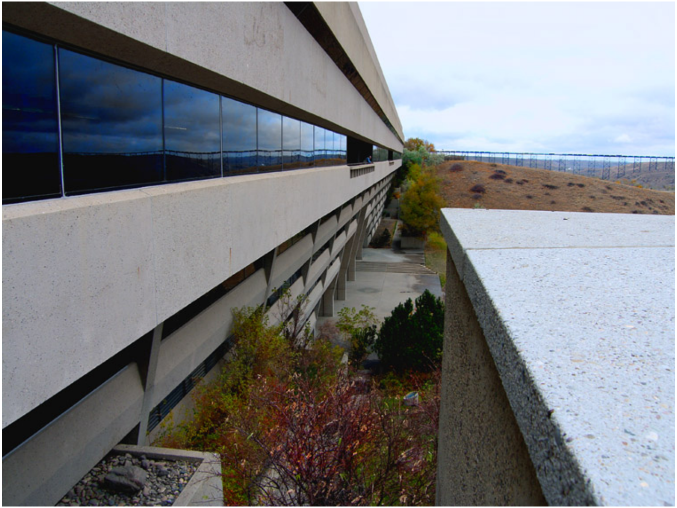
This method can be used to solve pairs of congruences

$$\begin{cases} p^2 & \equiv a \pmod{q}, \\ q^2 & \equiv b \pmod{p} \end{cases}$$

for other constants (a, b), e.g.,

- explicitly for $(\pm 2, \pm 2)$ or $(\pm 5, \pm 5)$,
- or “in principle" for any pairs (a, b).

Thank you



Lethbridge – Otto Rapp