New Proofs of the Irrationality of e^2 and e^4

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0. Prelude. The sequence $\{\alpha_n\}_{n=1}^{\infty}$ –with initial terms 0, 1, 1, 3, 3, 4, 4, 7, 7, 8, 8, 10, 10, 11, 11, 15,…–is not, perhaps, one of the best known sequences; it has **ID Number** A011371 and **Name** *n minus (number of 1's in the binary expansion of n)* in N. J. A. Sloane's wonderful On-Line Encyclopaedia of Integer Sequences (**[1]**). 2^{α_n} is the largest power of 2 dividing $n!$ (see section **5** for details); and some might like to think of α_n as being the number of 0's at the end of the binary expansion of $n!$.

The sub-sequence $\{\alpha_{2^m}\}_{m=1}^{\infty}$ will play a vital role in the new proof that I offer of the irrationality of e^2. In my proof it will be critical to have infinitely many n's for which α_n is *almost n* (more precisely that α_n differs from n by a bounded amount), and that will be achieved with the choice $n = 2^m$, which gives $\alpha_n = n-1$; in contrast it should be noted that $n = 2^m - 1$ gives $\alpha_n = n - m$, with α_n differing from n by increasingly larger amounts as m increases[1].

1. Introduction. In their lovely *Proofs from THE BOOK*[2] **[2]**, Aigner and Ziegler remark of the classic proof (revisited below) of the irrationality of e:

> This trick [multiply-by-n-factorial], however, *isn't even good enough* [my emphasis; see section **3** later for an elaboration of their thinking on this point] to prove that e^2 is irrational (which is a stronger statement). For this we need a different method…

Perhaps it is not well known that there *is* a proof by C. L. Siegel (p. 4 and 5, **[3]**) of the irrationality of e^2 which successfully uses the multiply-by-n-factorial trick; indeed it's a proof that allows one to show that e is not a quadratic irrationality: $Ae^2 + Be + C \neq 0$, for all $A, B, C \in \mathbf{Z}$, $A \neq 0$.

In this note I give a new, elementary proof that e^2 is irrational, which also uses the classic multiply-by-n-factorial trick (but in a manner different from Siegel); in fact my idea, married to Siegel's, enables one to show that e^2 is a quadratic irrational: $Ae^4 + Be^2 + C \neq 0$ for all $A, B, C \in \mathbf{Z}$, $A \neq 0$. Thus e^4 is irrational.

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[1] In fact it will be precisely for this reason that one should *not* 'multiply by *n factorial*' for *all n*, but rather by *n factorial*, for *selected n*.
[2] As they explain in their Preface, 'Paul Erdös liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, …'

I also note an uncountable set of irrational numbers whose irrationality can be established by using the idea in my e^2 proof, and I also pose some related questions.

2. A review of the standard *e* proof, and Siegel's e^2 proof (in a slightly altered form). The standard proof that e is irrational is well known: suppose e $(=e^1)$ is rational, then for some $a, b \in \mathbf{N}$

$$e = \frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + R_n(1) \tag{1}$$

Then

$$n!e = \frac{n!a}{b} = \left(n! + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \ldots + \frac{n!}{n!} \right) + n!R_n(1)$$

But $\dfrac{n!a}{b}$ is an integer for all $n \geq b,$ every term in the sum $\left(n! + \dfrac{n!}{1!} + \dfrac{n!}{2!} + \dfrac{n!}{3!} + \ldots + \dfrac{n!}{n!} \right)$ is an integer, and

$$0 < n!R_n(1) = n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots \right) = \left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots \right)$$
$$< \left(\frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \ldots \right) = \frac{1}{n}$$

giving

$$0 < n!R_n(1) < 1 \tag{2}$$

for all $n \geq 1.$ Thus (1) cannot hold for $n \geq b,$ since it leads to $\alpha = \beta + \gamma,$ $\alpha, \beta \in \mathbf{Z},$ and $0 < |\gamma| < 1$ – a classic type of impossibility in irrationality and transcendence theory. It follows that e is irrational.

Now suppose $Ae^2 + Be + C = 0,$ for some $A, B, C \in \mathbf{Z},$ $A \neq 0.$ Then $Ae + B + Ce^{-1} = 0,$ giving

$$A\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + R_n(1) \right) + B + C\left(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!} + R_n(-1) \right) = 0,$$

and

$$n!A\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \right) + n!B + n!C\left(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!} \right) \text{ [all three are integers]}$$
$$+ n!AR_n(1) + n!CR_n(-1) = 0 \tag{3}$$

But

$$|n!AR_n(1) + n!CR_n(-1)|$$
$$= \left| A\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots \right) + C\left(\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \ldots \right) \right|$$
$$\leq |A|\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots \right) + |C|\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots \right)$$

$$\leq (|A|+|C|)\left(\frac{1}{(n+1)}+\frac{1}{(n+1)^2}+\ldots\right)=\frac{|A|+|C|}{n}<1 \text{ for } n>|A|+|C|$$

and from (3) $n!AR_n(1)+n!CR_n(-1)\equiv 0$ for $n>|A|+|C|$.

Thus $AR_n(1)+CR_n(-1)\equiv 0$ for $n>|A|+|C|$. But since $R_n(1)$ is positive, and $R_n(-1)$ is alternately positive and negative, then A and C must be identically 0, contrary to $A\neq 0$. Thus e is not a quadratic irrationality.

3. The obvious attempt at extending the standard proof to e^2 fails[3]. Suppose e^2 is rational; then for some $a,b\in\mathbf{N}$:

$$e^2=\frac{a}{b}=1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\ldots+\frac{2^n}{n!}+R_n(2) \tag{4}$$

Then

$$n!a=\left(bn!+\frac{bn!\times 2}{1!}+\frac{bn!\times 2^2}{2!}+\frac{bn!\times 2^3}{3!}+\ldots+\frac{bn!\times 2^n}{n!}\right)+bn!R_n(2)$$

(5)

Now, however, we do not get an *immediate* impossibility as before; for although $\left(bn!+\frac{bn!\times 2}{1!}+\frac{bn!\times 2^2}{2!}+\frac{bn!\times 2^3}{3!}+\ldots+\frac{bn!\times 2^n}{n!}\right)$ is an integer–as is also $n!a$–now, however, $bn!R_n(2)$–which is an *integer* because the difference of the other two terms is one–does *not* lie between 0 and 1 for sufficiently large *n*. In fact $bn!R_n(2)\to\infty$ as $n\to\infty$ since

$$\frac{b2^{n+1}}{n+1}<bn!R_n(2)=\frac{b2^{n+1}}{(n+1)}+\frac{b2^{n+2}}{(n+1)(n+2)}+\ldots=b2^n\left(\frac{2}{(n+1)}+\frac{2^2}{(n+1)(n+2)}+\ldots\right)$$

$$<b2^n\left(\frac{2}{(n+1)}+\frac{2^2}{(n+1)^2}+\ldots\right)=2^n\frac{2b}{n-1} \text{ (for } n>2)$$

Thus

$$\frac{b2^{n+1}}{n+1}<bn!R_n(2) \tag{6}$$

and

$$bn!R_n(2)<\frac{b2^{n+1}}{n-1} \text{ (for } n>2) \tag{7}$$

4. The simple idea that leads to a new proof of the irrationality of e^2, a small impediment, and its resolution: Do *not* abandon (5) because of the unhelpful (6); instead consider the following:

[3] This is the point of the remark by Aigner and Ziegler.

- Not only are $n!a$ and $\left(bn!+\frac{bn!\times 2}{1!}+\frac{bn!\times 2^2}{2!}+\frac{bn!\times 2^3}{3!}+\ldots+\frac{bn!\times 2^n}{n!}\right)$ integral, but both are divisible by 2^{n-1} for *suitably chosen* values of *n* (that is a delicate point to which we will shortly turn our attention)
- $bn!R_n(2)$, is positive, and bounded above by $\frac{b2^{n+1}}{n-1}$, namely 2^n times $\frac{2b}{n-1}$, and the *latter* tends to 0 as *n* tends to infinity

Then, dividing through (5) by 2^{n-1} leads to $\alpha = \beta + \gamma,\ \alpha, \beta \in \mathbf{Z}$ (and thus $\gamma \in \mathbf{Z}$) with

$$0 < \gamma = \frac{bn!R_n(2)}{2^{n-1}} < \frac{1}{2^{n-1}}.\frac{b2^{n+1}}{n-1} = \frac{4b}{n-1} < 1 \text{ for } n > 4b+1$$

and thus γ cannot be integral for $n > 4b+1$, proving[4] that e^2 is irrational.

That analysis depends on $\frac{n!2^r}{r!}$ (for every *r* in the range $0 \le r \le n$) being divisible−for $n = 2^m$ $(m = 1, 2, 3, \ldots)$ − by an integral power of 2 which is *at least* 2^{n-1}, a detail to which we now turn our attention.

5. The *exact* power of 2 dividing *n*!, and a *minimum* power of 2 dividing $\frac{n!\,2^r}{r!}$ $(0 \le r \le n)$. First, it might be worthwhile to see some explicit exponents (see the **Maple** worksheets e^2 and e^4.mws (**[4]**, also in html format for any reader who doesn't have Maple) I have created to accompany this paper).

- For $n = 15$, the exponents of the largest powers of 2 dividing
  $$\frac{n!2^r}{r!}\ (0 \le r \le n) \text{ are}$$
  11, 12, 12, 13, 12, 13, 13, 14, 12, 13, 13, 14, 13, 14, 14 and 15
- For $n = 16$, the exponents of the largest powers of 2 dividing
  $$\frac{n!2^r}{r!}\ (0 \le r \le n) \text{ are}$$
  15, 16, 16, 17, 16, 17, 17, 18, 16, 17, 17, 18, 17, 18, 18, 19 and 16

As is well known (see almost any text on elementary Number Theory), if *p* is any prime, and $p^{\alpha_{n,\,p}}$ is the largest power of *p* dividing $n!$, then $\alpha_{n,\,p}$ is given by

$$\alpha_{n,\,p} = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

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[4] Subject to detail about being divisible by 2^{n-1} being completed.

where $[x]$ is the integer part of x. In particular, letting $p = 2$, and 2^{α_n} be the largest power of 2 dividing $n!$, we have

$$\alpha_n = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \ldots \qquad (9)$$

The fact that α_n is less than n is seen by dropping the fractional parts in (7), since we have

$$\alpha_n = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \ldots \le \frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \ldots < n$$

and that $\alpha_n = n-1$ for $n = 2^m$ $(m = 1, 2, 3, \ldots)$ is seen from

$$\alpha_n = \alpha_{2^m} = \left[\frac{2^m}{2}\right] + \left[\frac{2^m}{2^2}\right] + \ldots = 2^{m-1} + 2^{m-2} + \ldots + 1 = 2^m - 1 = n - 1$$

Also, if one sets $n = 2^m - 1$ $(m = 1, 2, 3, \ldots)$ then

$$\begin{aligned}\alpha_n = \alpha_{2^m} &= \left[\frac{2^m - 1}{2}\right] + \left[\frac{2^m - 1}{2^2}\right] + \ldots + \left[\frac{2^m - 1}{2^{m-1}}\right] \\ &= (2^{m-1} - 1) + (2^{m-2} - 1) + \ldots + (2^1 - 1) \\ &= (2^m - 1) - (m - 1) = n - m\end{aligned}$$

and thus α_n can differ from n by arbitrarily large amounts. It is precisely for the latter reason that one should *not* multiply throughout (4) by *general n*, but rather by *suitably chosen* values of n: those that are powers of 2.

6. e^2 is not a quadratic irrationality. Suppose $Ae^4 + Be^2 + C = 0$, for some $A, B, C \in \mathbf{Z}$, $A \neq 0$. Then $Ae^2 + B + Ce^{-2} = 0$, giving

$$A\left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \ldots + \frac{2^n}{n!} + R_n(2)\right) + B + C\left(1 - \frac{2}{1!} + \frac{2^2}{2!} + \ldots + \frac{(-2)^n}{n!} + R_n(-2)\right) = 0,$$

and $n!A\left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \ldots + \frac{2^n}{n!}\right) + n!B + n!C\left(1 - \frac{2}{1!} + \frac{2^2}{2!} + \ldots + \frac{(-2)^n}{n!}\right)$ [all three are integers]

$$+\, n!AR_n(2) + n!CR_n(-2) = 0 \qquad (10)$$

But

$$\begin{aligned}&\left|n!AR_n(2) + n!CR_n(-2)\right| \\ &= \left|A\left(\frac{2}{(n+1)} + \frac{2^2}{(n+1)(n+2)} + \ldots\right) + C\left(\frac{(-2)^{n+1}}{(n+1)} + \frac{(-2)^{n+2}}{(n+1)(n+2)} + \ldots\right)\right|\end{aligned}$$

$$\le |A|\left(\frac{2}{(n+1)}+\frac{2^2}{(n+1)(n+2)}+\dots\right)+|C|\left(\frac{2}{(n+1)}+\frac{2^2}{(n+1)(n+2)}+\dots\right)$$
$$\le (|A|+|C|)\left(\frac{2}{(n+1)}+\frac{2^2}{(n+1)^2}+\dots\right)=\frac{|A|+|C|}{n}<1 \text{ for } n>|A|+|C|$$

and (3) implies that $n!AR_n(2)+n!CR_n(-2)$–and thus $AR_n(2)+CR_n(-2)$–is *identically zero* for all sufficiently large n. But since $R_n(1)$ is positive, and $R_n(-1)$ is alternately positive and negative, then A and C must be identically 0, contrary to $A \neq 0$. Thus e^2 is not a quadratic irrationality.

Theorem. Let $\{a_n\}$ be any bounded infinite sequence of natural numbers, then $\sum_{n=0}^{\infty}\frac{a_n 2^n}{n!}$ is irrational (and, of course, so also is $\sum_{n=0}^{\infty}\frac{a_n}{n!}$, though that is not a novel observation)

7. Comments and some questions. There are obvious refinements that one could make of this theorem, but they would be rather artificial; for example one could replace *natural numbers* with *integers*, with the added proviso that $a_n \neq 0$ for infinitely many $n=2^m$, or one could relax the boundedness condition, and allow some modest growth bound on the sequence $\{a_n\}$ along the lines of $a_n = o(\log_2 n)$ as $n \to \infty$.

Of course this theorem produces an uncountable number of irrational numbers, and it is highly likely that each of them is not only irrational, but is, in fact, transcendental. Thus it might be of interest to provide an answer to a question like: does $\sqrt{2}$ have no representation of the form $\sum_{n=0}^{\infty}\frac{a_n 2^n}{n!}$ subject to the above conditions?

One would like to have an elementary proof that e^3 is irrational…

REFERENCES

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3. C. L. Siegel, *Transcendental Numbers*, Princeton University Press, 1949.

4. J. Cosgrave, related Maple worksheet in Public and Other Lectures section, http://www.spd.dcu.ie/johnbcos

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