REPORT ON NUMBER THEORISING WITH TALENTED YOUTH

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General Introduction

Dublin City University has founded, in collaboration with the Centre for Talented Youth at the John Hopkins University in the U.S.A, the 'Irish Centre for Talented Youth', whose aim is to cater for the needs of young people with an identified mathematical/scientific/other talent.

In July '93 the D.C.U. Centre launched its first Summer programme, of three weeks duration, which was attended by about 180 young people (attendance was determined by performance in the Scholastic Aptitude Test, the most widely used test of college entry in the U.S.A.) from Ireland and the U.S.A., taking courses in Drama, International Affairs, Bio–Technology, Archaeology, Media and Communications, Computers, Mathematical Models and 'Investigational Mathematics'.

There were 31 students, all from Ireland, studying mathematics. They were divided into two groups (according to their SAT scores); one, a group of 15, was taught by a highly experienced and enthusiastic secondary school teacher (assisted by a former student of his) called Martin Hilliard; the other group of 16 was tutored by myself for the first week (and one day in the final week), and by Professor Alastair Wood (Head of the School of Mathematics in DCU) and a former student of his, Dr. Fiona Lawless, for the second and third weeks.

My group and I met every day for five hours; three hours every morning (with two brief breaks) and immediately after lunch for two hours (with one short break). My group consisted of three girls and thirteen boys, and they came from all corners of Ireland. One was aged 12, three aged 13, five aged 14, six aged 15 and one aged 16, and only one had started the 'Leaving Certificate' programme in Secondary (High) School. As to the number of years of secondary schooling they had completed, four had completed 1 year, six 2 years, two 3 years and four 4 years.

They were a perfectly normal group of individuals, only unusual in that they were (by and large) possessed of a passion for learning – (a D.C.U. Mathematics postgraduate student, Shane O'Dowdall, sat in on almost all of my classes with them, acting as my diarist. On our first morning, after three and a half hours of solid work, during which we had only had one short break, he had to call my attention to the time and suggest we take a break for lunch. As one of my own teachers used to say, "how quickly time flies when one is thoroughly enjoying one's self!") – and working with them has been one of the most, if not the most, satisfying experiences of my twenty five years of attempting to teach Mathematics.

It would not be possible to record everything that passed between us, and so in this report I wish only to record some of the early work that they produced, before finally describing what I consider to be the most substantial body of work that we managed to produce together. I have chosen only that early work which relates directly to the latter.

My ideal reader of this report is a mathematician with an interest in teaching, but I would also wish it to be of interest to non–mathematicians, especially those who mistakenly think that Mathematics is all about unmotivated formulae. I have had the latter very much in mind writing this report.

Mathematical Introduction

When I first met my group I stressed that the work I hoped to do with them – 'Elementary' Number Theory (whose deceptively simple sounding questions have attracted the attention of some of the greatest mathematicians (Euclid, Fermat, Euler, Lagrange, Gauss, …), but which, regrettably, doesn't form part of their mathematical education in school) would not require that they had already covered a certain body of mathematical work; and thus they wouldn't need to know anything about, for example, trigonometry, co-ordinate geometry, Calculus,… ; they would *only* need to be able to add, subtract, multiply and divide, have innate mathematical ability, and be interested in thinking about mathematical questions.

I also said that I would avoid the use of jargon and would use only simple language (of course I intended to expand their mathematical vocabulary, but only when the need for it arose in a natural way). Most importantly I told them that they would not be passive learners of Mathematics (any fears that I might have had about that were very quickly dispelled!), that the way in which I hoped to make progress was not to set about teaching them anything specific, but rather to ask them some questions, see how they coped with them, and see where we got to. In my wildest dreams I could not have imagined that they would come up with as much as they did. Of course I will record their weaknesses as well as their strengths, though the latter completely outweigh the former.

I will only record at this early stage (without giving too much away) that some of them managed to discover for themselves the classic proofs of the irrationality of numbers like $\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{6},\ldots,\sqrt[3]{2},\sqrt[3]{3},\sqrt[3]{4},\ldots$, *etc.*; also the structural connections (with proofs, after some initial help) between what I call 'L–'and 'R–' approximations (namely, the best possible left and right rational approximations) to $\sqrt{2}$, and corresponding results for other similarly endowed numbers: $\sqrt{5},\sqrt{10},\sqrt{13},\sqrt{17},\sqrt{26},\ \sqrt{29},\sqrt{37},\ldots$); a proof that $\sqrt{3}$ has no L– approximations (and its extensions to other similar numbers: $\sqrt{6},\sqrt{7},\sqrt{8},\sqrt{11},\sqrt{12},\sqrt{14},\sqrt{15},\sqrt{18},\sqrt{19},\ldots$); the connection (with proof) between the R–approximations to $\sqrt{3}$ (and similar results for other numbers which don't have L–approximations, but which do have R–approximations).

In the following, whenever I use a term or notation, the reader should understand that it had already been introduced in an earlier discussion which I have not described in this report. Finally, '**JC**' stands for myself, '**S**' for someone or several, and '**RM**', '**EF**', etc. for individuals in my group. For the latter I give their full name when they make their first contribution.

An apology and explanation

When I first sat down to type this report I did not realise that it would turn out to be as long as it proved to be. So, I apologise for its length and I ask a reader to accept this explanation: if I were to read such a report by a colleague from another country I would be most unhappy to read something like: "I asked them a series of questions as a result of which they managed to prove that various numbers like, $\sqrt{2}, \sqrt{3}$ etc. were irrational and they also managed to prove results about best possible rational approximations to such numbers, etc., etc."

I would want to know: what questions were asked? Leading ones maybe? What kinds of responses were given? Also, it's very pleasing to get good responses, but how were poor responses dealt with? How was new terminology introduced? It is my belief that I have answered these, and other questions in this report.

Monday 12th July 1993

JC: Does anyone know what an 'abundant' number is? Or a 'deficient' number? (No one knew; I didn't expect that anyone would, and I continued:) Had anyone ever heard of a 'perfect' number?

Richard Murphy: (a 15–year old boy who had been coached for, but not made it into, the Irish Olympiad team): A perfect number is a number whose factors add up to itself.

JC: All of its factors?

RM: No, not all of them; all of them apart from the number itself.

JC: Good. Can you give me an example?

RM: Twenty eight; its factors are one, two, four, seven, fourteen and twenty–eight, and one, two, four, seven, and fourteen add up to twenty–eight.

JC: If I let out that 'abundant' and 'deficient' had something to do with 'perfect', what do you think they might mean?

S: 'Abundant' means the sum of all the factors (excluding the number itself) is more than the number; 'deficient' means less.

JC: Good. Can you give me some examples?

S: 12 is abundant, factors are: 1, 2, 3, 4, 6, (12); $1+2+3+4+6>12$.
14 is deficient, factors are: 1, 2, 7, (14); $1+2+7<14$.

Then completely out of the blue:

Elanor (correct spelling) Foley: (a 15–year old girl): Is every multiple of an abundant or deficient number also an abundant or deficient number?

JC: What do you mean exactly? Do you mean what you have just said (and I repeated as above) or do you mean 'is every multiple of an abundant number also abundant and every multiple of a deficient number also deficient?'

EF: I mean the last thing you said.

JC: Is this true?

They tried some abundant examples and it worked for those, and as no one could prove it always worked (it was only the first day, and they would not have been familiar with what constituted a 'proof' in Number Theory. They seemed to have an innate appreciation that this *needed* proving, unlike almost all of my own regular degree students who think that something is true if it is seen to 'work' without fail for several cases) I suggested that they think about it overnight. (A reader who is not familiar with this might like to try to prove it).

(The next day **R** presented a proof – not perfect in its presentation – but which showed that he understood the *essential* idea. None of my own 'third level' students have ever been able to prove it for me, and even when I have illustrated the key idea for the particular case of all multiples of 21, none have ever been able to argue the general case).

JC: What about the deficient case?

S: 14 is deficient and 2 time 14 is 28, is perfect.
14 is deficient and 3 times 14 is 42, is abundant (checked).
The same for 4, 5 and 6 times 14 (checked).

JC: Maybe they're all abundant apart from 2 times 14?

S: I think 14 squared is deficient.

JC: Think or know? Have you checked it? Why 'squared'? (As I was getting no response to these questions I just said:) OK Let's check it. The factors of 'fourteen squared' (written 14^2) are what?

S: 1 and 196 (the 'co-factor' of 1), 2 and 96 (the co-factor of 2), 4 and 48…

JC: (seeing and seizing a golden possibility): Why have you not said '3 and ''? How do you know that 3 isn't a factor of 196?

In asking such a question you have no idea if it will lead to anything or not. If someone had just said: "196 isn't divisible by 3 because when you 'divide 3 into 196' you get a remainder of 1", they would have been giving a factually correct reply, could have thought I was perhaps being a little pedantic, and it wouldn't have led to anything in particular.

RM: 196 is not divisible by 3 because the sum of its digits is not divisible by 3.

Off on a tangent: I was, of course, absolutely delighted to get such a reply. I asked if there were any of them for whom **R**'s reply meant something, and for some, but not all, it did. Because it had come up in this natural way, I decided to pursue the question of determining divisibility by digital considerations. I will not give a detailed account (as this is a very routine and elementary topic), and so I will only record the points that were discussed and emphasised:

(i) That **R**'s (correct) reason was directly related to 'base 10' representation.

(ii) That other rules obtained for divisibility by 2, 4, 5 and 8.

(iii) That, for other bases, the rules already discovered for base 10 couldn't be just changed without thought; that, for example, the rule that 'a whole number is divisible by 2 if (and only if) its final digit (in a base) is divisible by 2' isn't true for any base. Rather it depends on the base; and so, for example, $(17)_9$, although divisible by 2, does not have final digit divisible by 2.

(iv) With regard to divisibility by 6, someone had (correctly) suggested that 'a whole number is divisible by 6 if it is divisible by 2 and also by 3', and I asked would it be correct to say, for example 'a whole number is divisible by 10 if it is divisible by 2 and 5' (I was told 'yes'), and other variations of this. But how about, say, divisibility by 24? Would it be correct to say that a whole number is divisible by 24 if it is divisible by 4 and 6? They didn't fall for that one (12 is divisible by 4 and 6, but not by 24; and other variations). So, how could they be so certain about the 6 case?

Return to RM's comment about 196: We eventually went back to 196 and found all of its factors (and they were able to say, for example, 5 is not a factor as the last digit is not 0 or 5, 6 is not a factor as 3 is not a factor, etc.): 1, 2, 4, 7, 14, 28, 49, 96, 196 and so 196 is abundant since $1+2+4+7+14+28+49+96=201>196$.

So, we still hadn't resolved the question about proper multiples of 14, that maybe, apart from 2 times 14, they are all abundant. Then, after our only break of that first morning:

Bryan O'Higgins: (a 13–year old who had completed *only one* year at High/ Secondary school): I think I have an example where it's deficient for 14.

JC: Namely?

BH: 11 times 14.

We all found it's factors, namely: 1, 2, 7, 11, 14, 22, 77, 154 and 154 is deficient since $1+2+7+11+14+22+77=135<154$.

JS: Good! Now why did you pick on 11 times 14? Trial and error.

BH: I picked on '11' because it was a prime, and it seemed to me that if I picked on a prime I would be able to keep down the number of factors and maybe get a deficient number.

That (for his age) was an incredibly insightful (though not quite correct) remark!

JC: What about other prime multiples of 14? (Remember we had already encountered 2 times 14, 3 times 14, and 5 times 14). What about 7 times 14?

We collectively checked that it too is deficient. For a moment I considered posing this question: 'could you prove that if p is any prime, and $p > 5$ then $14p$ deficient?' but I was keen to explore potentially richer ideas and so proceeded:

JC: It's true that 196 is not divisible by 3 for the reason that **RM** gave, but I would like to put to you another reason for that, *which is completely independent of* '*base*' (this was one of the rare occasions that I decided to tell them something, as left on their own it almost certainly would not have occurred to any of them). Can anyone prove that if a whole number is *not* divisible by three, then the square of that whole number is *not* divisible by 3? And could anyone give an example to show that I couldn't replace '3' with a '4' and still make the same claim?

At first no one could prove the first claim, but I was given examples like: 4 does not divide 6 but 4 does divide 6^2, 4 does not divide 10 but 4 does divide 10^2, etc.

Now I wish to jump ahead to the following day.

Tuesday 13th July 1993

Before I had time to draw breath:

RM: I can prove that if a whole number is not divisible by 3 then it's square is not divisible by 3, and that every multiple of an abundant number is abundant.

I intend only reporting on the first of these, as it eventually leads to the main work that they produced.

JC: Let's take the first of those; how did you do it?

RM: Using mods.

Anyone familiar with congruences/modular arithmetic will realise that that's what he had in mind (he had been coached on that in connection with preparation for the Irish Olympiad), but I wasn't keen to have him talking about 'mods' and possibly mystify many in the class, especially the younger ones. I preferred instead to try to get him to use the simpler and more familiar language of remainders on division by 3, so I proceeded:

JC: I know what you mean, but the language of 'mods' is perhaps not familiar to everyone. I wonder if you could perhaps strip away the jargon of 'mods' and give me your solution, but expressed in simpler, more familiar language? My own personal preference is not to use technical, possibly unfamiliar terms, when simpler terms are available?

RM: (He had absolutely the correct idea, and it was only a case of my using a guiding hand to express it, using notation already established in class, as follows:)

Claim: If $b \in \mathbf{Z}$ and $3 \nmid b$ then $3 \nmid b^2$.

Proof: Since $3 \nmid b$ then $b = 3n + 1$ or $b = 3n + 2$ for some $n \in \mathbf{Z}$.

Then $b^2 = 9n^2 + 6n + 1$ or $b^2 = 9n^2 + 12n + 4$, and so

$b^2 = 3(3n^2 + 2n) + 1$ or $b^2 = 3(3n^2 + 4n + 1) + 1$, and thus

$b^2 = 3B + 1$, some $B \in \mathbf{Z}$. Thus $3 \nmid b^2$.

JC: Good. I hope everyone sees that it's really quite simple. It's just a case of looking at the integer you have, dividing it by 3, seeing what the remainder is (and we had already had a discussion about the meaning of 'remainder'; that if, for example, one wrote: $29 = 3.8 + 5$, one *could* say that 29 leaves remainder 5 on division by 3, but that by rewriting: $29 = 3.8 + 3 + 2 = 3.(8 + 1) + 2 = 3.9 + 2$, then *more naturally*, 29 leaves remainder 2 on division by 3) and then seeing how that *influences* the remainder that its square leaves on division by 3. (pause). Now, I wonder if you can tell me some other numbers which behave like three?

They quickly gave me examples of ones that behave like 3 (2, 3, 5, 6, 7) and one's that don't $\{4$ (already known), 8, 9$\}$. Then:

JC: (wrote on board:) Which values of a have/don't have the property that whenever $a, b \in \mathbf{Z}$ and $a \nmid b$, then $a \nmid b^2$?
Have the 'property': 2, 3, 5, 6, 7, 10.
Don't have the property: 4, 8, 9.

EF: I think that multiples of four don't have the property.

JC: Good! So the 'don't have' list might look like this?:
4, 8, 9, 12, 16, 20, …
How can you express your guess?

EF: Four *m*.

JC: How can you prove it?

EF: Take *b* to be the half of *a*.

JC: Good! I'd like to write up a formal proof (and wrote).

Simple Observation : For all $m \in \mathrm{N}$, $a = 4m$ hasn't the 'property'.

Proof : Let $b = 2m$, then $a \nmid b$ since $4a \nmid 2a$.

But then $b^2 = (2m)^2 = 4m^2 = 4m.m = a.m$.

Thus $a \mid b^2$ and so *a doesn't* have the property.

JC: Can anyone think of any other values which don't have the property?

BH: Multiples of nine.

JC: Good! Now would everyone please put their name at the top of a piece of paper and write up for me a proof that multiples of 9 don't have this property, and hand it up to me as I would like to read it later.

I allowed only about two minutes for everyone to do this. I didn't want to use up valuable class time by looking at their work then; I put it aside to look at that evening. I will only comment briefly on their efforts (I will send photocopies of their work to anyone who is seriously interested, with the students' names hidden to respect their privacy): one of them didn't know what to do, one offered as a proof the mere verification of a single example ("e.g. $a = 18$, $6^2 = 36 = 18x$), one made a correct choice for 'b' but then went astray, one initially took 'b' to be 4½m, then changed it to $9m$ and got nowhere, one just wrote that $a = 9m$ gave $9a = 81m$ and made no choice of an appropriate 'b', but the other eleven (including all four who had completed only one year at secondary/high school) gave correct or essentially correct proofs. I will just record the work of one of the latter, a 13–year old girl (**Roisín Loughran**, with only one year of secondary schooling completed):

Proof : Let $b = 3m$, then $a \nmid b$

but $b^2 = (3m)^2 + 9m^2 = 9m(m) = am.$, i.e. $a \mid b^2$.

One might quibble that she hadn't '$\in$ **N**' in appropriate places, but that really would be a bit pedantic (I remarked on that when I later had an opportunity to speak with her about her work).

When I had collected their work I invited a proof from whoever could provide one, and after a discussion {during which it emerged that besides the choice of $3m$ for 'b', one could also choose – though there would be no need to do so – other numbers like $6m$, $12m$, $15m$, $21m$, … (but *not* $9m$, $18m$, $27m$, …) for 'b'} I wrote up a proof modelled on the '$4m$' case above. As soon as I had finished:

BH: I think that any number that's a multiple of a square doesn't have the 'property'.

JC: So, six doesn't have the property because six is 'a multiple of a square', namely six is six times one squared?

BH: (protesting): No, no; the square has to be *bigger* than one.

JC: That's right! You see how *careful* you must be when you are making mathematical statements. So, can you all prove the following for me (wrote on board:)

Simple observation: Let $a \in \mathbf{N}$ such that $a = ms$, where $m \in \mathbf{N}$ and s is a square with
$s > 1$, then s will not have the 'property'.

I deliberately wrote it like this as I was curious to see how many, if any, would be sensitive enough to write in their proof *something like*: Let $s = S^2$, where $S \in \mathbf{N}$ and $S > 1$.

Once more I will comment briefly on their efforts: the five who hadn't been able to handle the '9*m*' case didn't appear to benefit from the above discussion (one started with "$a = 2,\ 2^2 = 4 > 1.\ a = 3m.\ a =$ is a square. $b = 4nc,\ b = ac,\ ac|b,\ ac|b$", another took '$b$' to be $4sm\ldots$, another just took a to be 9 and wrote that $9 \nmid 6$ but $9|6^2\ldots$) and many of the others now had difficulties.

The best effort (**RM**'s) was this:

Proof: Let $s = t^2$, then $a = t^2m$.
Let $b = tm$, then $a \nmid b$ (provided $b > 1$ as if $b = 1,\ a = b = m$ and $a|b$).
$b^2 = t^2m^2$ and so $a|b^2$, and so a does not have the property.

His "provided $b > 1$ as if $b = 1$" was, of course – as I had determined from speaking with him about it subsequently – an error made in the heat of the moment; in both places 'b' should, of course, be 't'.

Another good effort came from a 16–year old girl (**Clare Kelliher**, and the only one in the group who had started the Leaving Certificate programme) who wrote:

Proof: $a = sm$. Let $b = \sqrt{s}.(m)$, then $a \nmid b$ $(sm \nmid \sqrt{s}(m))$
but $a|b$, because $b^2 = (\sqrt{s}(m))^2 = sm^2$. i.e. $a|b = sm|sm$.

The minor faults are obvious, but her use of '$\sqrt{s}.(m)$' showed sensitivity and understanding (another student had chosen 'b' to be '$\sqrt{(sm)}$' throughout – how often does one have to comment to one's students: "be careful with bracketing!")

We had some further discussion about this 'property', particularly with regard to the more difficult part of it, namely, which numbers have the property. I will only record that they were acute enough to explicitly state that those numbers which have the property are precisely those which are not multiples of squares which are larger than one (I regret that I didn't introduce the standard term 'square-free').

Later, at the end of the afternoon session, I asked each of them to write their name on a piece of paper and tell me their age, how many years of schooling they had had, and let me know how they had found the work so far. I was, of course, going to give more credence to someone who told me they were having difficulties, then to someone who maybe said they found it all too easy, when perhaps there was no evidence for that. My offer concerning photocopies also applies to these responses, but I will just briefly list a range of them here:

"I find it sometimes hard to understand the proofs in class, but I can understand them better later… I am enjoying the course and the method of teaching." (a 14–year old)

"The thing I find difficult are writing proofs. I think everything else is fairly easy." (a 15–year old)

"The question of the 'property' was the hardest so far and I found it fascinating." (a 15–year old)

"What I find difficult is writing out proofs." (a 14–year old)

"I have found the work both interesting and definitely challenging having never come into contact with number theory before. I am enjoying the new subject and feel that I am coping adequately. The idea of proving an argument explicitly is also new and is perhaps the most difficult area. But I have enjoyed learning the technique." (a 15–year old)

"It is the first time I have done proofs of this sort so it was hard at first to understand but I am beginning to now." (a 14–year old)

Wednesday 14th July 1993

I was keen to see how they would cope with 'irrational numbers', but I didn't give any indication that this was my aim. Having asked if they all know what 'square root' means, and tested that they did know by asking them to tell me the values of $\sqrt{9}, \sqrt{4}, \sqrt{2.25}$, etc., I proceeded:

JC: Would you tell me the value that your calculators display for $\sqrt{7}$? (I will explain later why I quite deliberately choose $\sqrt{7}$ rather than, say, $\sqrt{2}$).

S: (and I displayed on the board): $\sqrt{7} = 2.645751311$.

JC: Could that be the exact value of $\sqrt{7}$, just as 1.5 is the exact value of $\sqrt{2.25}$ and if it isn't, might it be that if we had a more accurate calculator or computer, then maybe by going out to – say – 50 decimal places we might get the exact value? I'm really asking if the square root of seven has a decimal expression which comes to an end.

After some moments, during which it seemed that at least one of the group thought that the above displayed value was the exact value (simply because that was the value churned out by the calculator), I got this contribution:

RM: The square root of seven doesn't have a decimal value that ends. (I asked why, and he continued:) Well if it did and ended in a one then its square would end in a one and so wouldn't be seven, and if it ended in a two it's square would end in a four.

I quickly intervened at that point and picked out individuals in the group and asked: "and if it ended in a three, four, five, … , its square would end in what?" I got correct replies from those I asked, apart from one who couldn't tell me in the case of 'six'. So we had a proof (of course I praised **RM** for coming up with the idea) – not formally written up – that $\sqrt{7}$ doesn't have, what I deliberately called, a 'terminating decimal expansion'. I mentioned that numbers that have terminating decimal expansions are just 'rational numbers' (of course I stated precisely what that meant: a number is rational if it can be expressed as the ration of two whole numbers, or, as is more commonly called by mathematicians 'integers') whose denominators happen to be 1, 10, 100, 1000 etc.; in other words just 'powers of ten', and whose numerators happen to be whole numbers. Then:

JC: Might it be possible for there to be a rational number which was the exact value of the square root of seven?

[**Aside**: I had deliberately picked on $\sqrt{7}$ rather than $\sqrt{2}$, because I was concerned that someone in the class might have already heard of, or been coached in, 'irrational numbers'; and if so then it would have been most likely that it would have been $\sqrt{2}$ (how many undergraduate students of Mathematics are there who, if they have encountered irrational numbers at all, have only met the one example $\sqrt{2}$?). Of course, had that been so, I would have been disappointed in them if they didn't then just say: "$\sqrt{2}$ is an irrational number and the proof of that can be easily altered to show that $\sqrt{7}$ is also an irrational number".]

JC: Would you please use your calculators to tell me the decimal values of the rational numbers $\frac{32257}{12192}$ and $\frac{514088}{194307}$ (I had worked these out in advance and had memorised them so that they didn't think that I was up to something if they saw me consult a piece of paper, and I wrote them up on the board, just under the already displayed approximate value of $\sqrt{7}$). When they called out the calculator values there was now displayed on the board:

$$\sqrt{7} = 2.645751311\ldots$$
$$\tfrac{32257}{12192} = 2.645751312\ldots$$
$$\tfrac{514088}{194307} = 2.645751311\ldots$$

I think that most of them were impressed with these. The middle one isn't $\sqrt{7}$ – if one is to believe one's calculator [later I spoke about the question of being careful about not putting too much faith in one's calculator, to always be thinking about what might be going on behind the scenes when one did a calculation. The most dramatic example I gave – which genuinely impressed them – was in connection with verifying Euler's demonstration that the sixth Fermat number, namely $F_5 = 2^{32} + 1$, which we had encountered in another earlier discussion – not recorded in this report – is divisible by 641. An 8–digit calculator will not only 'show' that 641 divides $(2^{32} + 1)$ but (absurdly) that each of $(2^{32} - 1)$, 2^{32}, $(2^{32} + 2)$ and $(2^{32} + 3)$ are divisible by 641 as well].

JC: So, there we have two rational numbers, and one of them is quite close in value to $\sqrt{7}$, and the other is even closer. Is the last one *so close* that it *might* in fact be *equal* to $\sqrt{7}$? Or maybe if we had a calculator with more than ten digit display and we re-calculated $\sqrt{7}$ and $\frac{514088}{194307}$ then we would find that they agreed on one more decimal place, but then differed on the next place? And so would not be equal. What I am really asking you is this: are there or aren't there integers a and b such that $\sqrt{7} = \frac{a}{b}$?

For the first time I had asked a question that met with no response, and after allowing some time to pass I continued:

JC: I don't wish to only ask this sort of question for $\sqrt{7}$, but also for other numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}$, etc. To take the first of these, $\sqrt{2}$, would you tell me your calculator values for $\sqrt{2}, \frac{47321}{33461}$ and $\frac{114243}{80782}$?

The following were then displayed:

$$\sqrt{2} = 1.414213562\ldots$$
$$\tfrac{47321}{33461} = 1.414213562\ldots$$
$$\tfrac{114243}{80782} = 1.414213562\ldots$$

JC: Might $\sqrt{2}$ be equal to one (or maybe both) of these rational numbers – the same kind of question as we were considering earlier – and, while we're at it, how about these two rational numbers? Are they or aren't they equal?

I got no contribution on the first question (and I should record that one of them thought these three numbers were all equal just because their calculator values agreed), but on the second:

RM: The two rational numbers couldn't be equal, because if they were then:

$$\tfrac{47321}{33461} = \tfrac{114243}{80782} = a \text{ (let's say), then } 47321 = 33461.a \text{ and } 114243 = 80782.a,$$

and from the first of these we get that a's decimal expansion must end in a '1', and that contradicts the other equation which would make $80782.a$ end in a '2', whereas it ends in a '3'.

JC: That seems like a good argument. Who accepts that?

ALL: We do.

JC: Well I certainly don't! **R**'s argument only *seems* to be OK, but it has a *big flaw* in it – *it rather takes for granted that a's decimal expansion has a last digit*, and I'm sure you can all give me some rational numbers whose decimal expansions don't terminate.

S: $\frac{1}{3} = 0.333\ldots$, $\frac{1}{6} = 0.166\ldots$, (JC:) $\frac{1}{7} = 0.142857$ (repeat).

EF: But wouldn't **R**'s argument work if you said let's take the last decimal digit of *a at infinity?*

I then embarked on a long monologue in which I tried to explain that there was no (meaningful) such thing as 'the *last* decimal digit of *a at infinity*' {if there was such an object, what would come just before it? What would it be called? The (infinity minus–one)th place? And the one before that? The (infinity minus–two)th place? And just before that?… and where would the 'join' be between these places. I suggested that it would be like trying to link two trains together, one of which had a first coach, a second coach, …, but no last coach, and the other one having a last coach, a second last coach,… , but no first coach}, at the end of which:

JC: Come on, surely someone can show me that $\frac{47321}{33461}$ and $\frac{114243}{80782}$ are equal or show me that they aren't.

Richard O'Callaghan: (a 15–year old boy who had completed three years at Secondary school): I think they're not equal. (**JC:** Why?).

RC: Because if they were, and you then subtracted them from each other you would get 0. (**JC:** So?) Well you would have (written on the board:)

$\frac{114243}{80782} - \frac{47321}{33461} = 0$. (**JC:** and?)

RC: Well if you then made a common denominator and carried out the subtraction, the numerator would be 114243 times 33461 minus 80782 times 47321, and that ends in a 1, so the difference couldn't be 0.

JC: Wonderful! At long last! (There was a great sense of relief in the class and some called out "Well done Robert!"). In fact not only does the numerator end in 1, but you are in for a big surprise! Calculate 114243 time 33461 minus 80782 times 47321 and see what you get!

ALL: (after a few moments calculation): It comes to just 1!

JC: Isn't that wonderful? So, these two rational numbers aren't equal, but they are *incredibly close* together, differing only by the *tiny* amount $\frac{1}{80782}$ times 33461. But now I would like to say this to you: Robert's proof is just perfect, but it can be more briefly expressed. How?

RC: Oh yes! Just cross multiply!

JC: Simple isn't it! Why didn't you say that ages ago?

We were just coming to the end of that third day's work and it was a great relief that someone had been able to come up with something, as I would have hated to have had to point out a reason myself.

JC: Now the question I would like you to think about overnight is simply this: are there or aren't there integers a and b such that $\sqrt{2} = \frac{a}{b}$? And the same question for other numbers like $\sqrt{3}, \sqrt{5}, \sqrt{6}$, etc.

Some written comments at the end of that day.

"I didn't understand how to get the answer to the question: 'do there exist $a, b \in \mathbf{N}$ such that $\sqrt{7} = \frac{a}{b}$?', but I was fascinated by the way you could form a fraction to be equal to the $\sqrt{2}$ (**JC comment**: obviously some misunderstanding there), and I hope we will soon be also able to do this." (a 14–year old)

"I feel myself compelled to argue against the frustration encountered regarding why $\frac{114243}{80782}$ and $\frac{47321}{33461}$ are not equal." (a 15–year old)

(**JC comment:** When the class was over I was talking with some of them about that very point, and I said that I could so easily have cut the whole thing short in class by saying: "Look, you're stuck, I'll tell you. Suppose they were equal; then cross multiplying you get two whole numbers equal to each other; but one of them ends in a 3, the other in a 2; that's impossible, so the two original numbers couldn't be equal. But I decided to let it run since I was getting some contributions. Those contributions were in error, but in my considered view a most important part of one's mathematical education is not just encountering arguments that 'work' but also ones that don't, and knowing why they don't work." Looking at those young faces, as I was saying that, I formed the impression that they understood why I had done what I had done, and as a teacher I could hardly ask for more.)

"It is difficult to understand the infinite decimal numbers and finding the square root of numbers other than squares of whole numbers." (a 14–year old)

"I found what we did today very interesting, and I was fascinated by the way the fractions were created. I eagerly anticipate when we are also able to form fractions, e.g. $\frac{47321}{33461}$ equal to $\sqrt{2}$ (**JC comment:** once again an apparent misunderstanding)." (a 14–year old)

Thursday 15th July 2003

Each day has its own rewards and excitements – if the latter hasn't come across it's because I'm no playwright! – but for me it was the last two days of the first week, and the extra day that I had in the third week, which were so thrilling. They produced so much work of *real* value in such a short time, at times at breakneck speed. As soon as we started:

BH: You can't have $\sqrt{2} = \frac{a}{b}$ for integers a and b – (**JC:** why not?) – because if you did then you would get $b.\sqrt{2}$ and that is impossible. (**JC:** why?) Because $\sqrt{2}$ doesn't have a terminating decimal expansion, and you can't have a number which doesn't have a terminating decimal expansion (here $\sqrt{2}$) being multiplied by a whole number (here b) and get a whole number (here a)

I was so taken aback with this 'reason' that instead of asking the rest of the class what they thought of this, I just blurted out:

JC: No! no!, no! Not at all! If you take a number like, for example, $\frac{1}{6}$; it doesn't have a terminating decimal expansion: $\frac{1}{6}=.166666\ldots$; *but* if you multiply it by, for example, the whole number 12 you get the whole number 2. So your reason is just not acceptable. Yes? (He agreed.) Who can give me other similar examples?
I was given several examples and then:

RM: I don't think you can have integers a and b such that $\sqrt{2}$ equals $\frac{a}{b}$. (**JC:** why not?) Well if you did then a would be even (**JC:** why?)

RC: (jumping in) Because $\sqrt{2}$ has the 'property'.

JC: Good, good, I think I know what you mean, but can you just explain a bit more?

RC: Well, from $b.\sqrt{2} = a$ you would get $b^2.2 = a^2$ and so 2 would have to divide a as 2 divides a^2.

JC: That's good. I'd like to write that up in text book fashion (and I wrote on the board:)

Simple Observation : If there is a rational number which equals $\sqrt{2}$, then the denominator of that rational number must be even.

Formal Proof : Let $a,b \in Z$ with $\sqrt{2} = \frac{a}{b}$, then $b.\sqrt{2} = a$ and so we have $b^2.2 = a^2$, and so $2 \mid a^2$. But then since $a = 2A + 0,1$ for some $A \in Z$ we have $a^2 = 2A' + 0,1$ for some $A' \in Z$, and we have that when $2 \mid a^2$ it must be that $2 \mid a$.

(and wishing to take some initiative myself, but at the same time not give too much away, I continued:) So, if you have a rational number $\frac{a}{b}$ which equals $\sqrt{2}$ then the numerator, a, has to be even. *It has no choice in the matter*; it simply has to *be even*.

Yesterday we looked at the rational numbers $\frac{47321}{33461}$ and $\frac{114243}{80782}$, and we were wondering which, if any of them, might be equal to $\sqrt{2}$; now we can see at a glance that neither of them could possible be $\sqrt{2}$. Why is that?

S: Because the two denominators are odd.

JC: Exactly. If you have a rational number, and its numerator happens to be odd then it couldn't possibly be $\sqrt{2}$. It might be very close to $\sqrt{2}$ but it couldn't be $\sqrt{2}$. (pause). Now suppose you made slight alterations to the numerators of the above rational numbers to make new ones, but now with even numerators; let's say we alter the one that ends in a 1 by just changing the 1 to a 2, and alter the one that ends in a 3 by just changing the 3 to a 2. Now, you recalculate and compare with $\sqrt{2}$.

There was now displayed on the board:

$$\sqrt{2} = 1.414213562\ldots$$
$$\tfrac{47322}{33461} = 1.414243488\ldots$$
$$\tfrac{114242}{80782} = 1.414201183\ldots$$

JC: You see. Decent enough approximations. Not as good as those we've already seen. Certainly not equality. Incidentally, without using our calculators, how could we have told that these two rational numbers couldn't have been equal?

S: (yelled out): Cross multiply!

JC: Good; at least you've learned something! Of course it's only because of the endings being right. Now, Richard, you were telling me a while ago that '*a*' would have to be even. You were going to tell me something else?

RM: The '*b*' would have to be even also – (**JC:** Why?) – because (and I just wrote up on the board what he told me:)

with $b^2.2 = a^2 = (2A)^2 = 4A^2$, then $b^2 = 2A^2$ and so b is even.

JC: Good. And so if you were looking for a rational number which was equal to $\sqrt{2}$ then we would know that its numerator would have to be even – and so a rational number like, for example, $\frac{47321}{33461}$, couldn't possible be $\sqrt{2}$ as its numerator is odd – and if you went and altered it slightly and formed the rational number $\frac{47322}{33461}$ to make an even numerator so as to *at least give it a chance* of being equal to $\sqrt{2}$, that would also fall flat because of Richard's second observation, namely: if a rational number is equal to $\sqrt{2}$ then the denominator of that rational number must also be even. So *maybe* if we made another alteration and formed, say $\frac{47322}{33462}$, it might equal $\sqrt{2}$. *We couldn't immediately rule it out since now both numerator and denominator are even*. (pause). Quickly, use your calculators to see if it is $\sqrt{2}$. (Then was displayed:)

$\frac{47322}{33462} = 1.414201183\ldots = \sqrt{2}(= 1.414213562\ldots)$

JC: So, not equality. You've used your calculators to show that though. Suppose I picked the rational number $\frac{19606}{13862}$ (just a double alternative to $\frac{19601}{13860}$, already displayed on the board having the value $\underline{1.41421356}4$ ($= \sqrt{2} = \underline{1.41421356}2\ldots$); could you tell me if that is equal to $\sqrt{2}$ or not but without using your calculators. Pen and paper and everyone working on their own.

I allowed some minutes before collecting their work, which I will briefly record.

Their written responses to the question: is/isn't the rational number $\frac{19606}{13862}$ equal to $\sqrt{2}$?

- Three of them handed in very poor work; just to give one example: "This is not true, because when 13862 was 13861 looked like it might have been equal to $\sqrt{2}$, but now that the number has changed it has changed the answer of the division. It should no longer even look like $\sqrt{2}$."

- Four of them calculated – by long hand multiplication – the values of 19606^2 and 2.13862^2, and remarked that these were not equal, and so $\frac{19606}{13862}$ could not equal $\sqrt{2}$.

- Three remarked that 19606^2 ends in a 6, that 2.13862^2 ends in an 8, and thus that $\frac{19606}{13862}$ could not be $\sqrt{2}$.

- The remaining six gave quite perfect reasons ('perfect' because they went right to the heart of the matter, and found for themselves, without any hint whatever from me, the <u>key</u> to proving – given what has already been done – that $\sqrt{2}$ is an irrational number) for the rational number $\frac{19606}{13862}$ not being equal to $\sqrt{2}$. As four of these six (**EF**, **RM**, **BH** and **CK**) have already figured in this report so far, I would like to record in complete detail the work of the other two.

Eoin Bambury (a 14–year old boy who had completed two years at secondary school)

"Is $\frac{19606}{13862} = \sqrt{2}$?

No. This rational number cannot be equal to the $\sqrt{2}$, because when you divide top and bottom by two you get $\frac{9803}{6931}$. Here the numerator and denominator are both odd. This cannot possibly be the square root of 2 and we have already proven that for a rational number to be the $\sqrt{2}$, both numerator and denominator must be even." (**JC comment:** One might be pedantic and quibble with this, but it is quite clear that he understands.)

Kenneth Kearney (a 13–year old boy who had completed only one year at secondary school, and the one who had written at the end of the second day: "found maths hard at the start but I am gradually finding it easier", and at the end of the third: "I think I understood most stuff, but I am having a little difficulty in proving theories. But, thankfully, I am beginning to understand.") wrote:

" QUESTION: Is $\sqrt{2} = \frac{19606}{13862}$.

EXPLANATION: $\frac{19606}{13862} = \frac{9803}{6901}$. $\frac{19606}{13862}$ is not equal to $\sqrt{2}$ because when you simplify it down to $\frac{9803}{6931}$ which is equal to $\frac{19606}{13862}$, when you square $\frac{9803}{6931}$, you can't get 2 because the numerator and denominator are both odd.

PROOF: $\frac{19606}{13862} = \frac{9803}{6931}$, $\frac{9803}{6931} \neq \sqrt{2}$. $\therefore \frac{19606}{13862} \neq \sqrt{2}$. " [end of student quote]

I collected the work from them there and then, and asked who could settle the question that they had just been writing about. By discussion it was clear that the key thing to do was to 'reduce', and, if need be, to *repeatedly reduce* by a succession of division by 2 (obviously those who had already thought of this for themselves hardly needed telling).

Diversion The reader who is acquainted with the elementary first steps of the theory of irrational numbers can imagine how delighted I was at that point. Many of them – not realising (How could they? It was not in their vocabulary) that they had done so – had, by their own efforts, made up a proof that $\sqrt{2}$ is an irrational number.

Some of my readers might claim that this was only possible with such a group of students, and that the approach I have described above would not work with one's own regular students. I would agree with them, as I have tried it and I've had no success.

[With my own students I would have to let out that $\sqrt{2}$ is indeed an irrational number, and ask if they can come up with a proof of that (and the general observation that it is easier to settle something – when you know the outcome – comes into play). They can't, but a question *like*: "I wonder if any of you can find a reason – *apart from just slogging it out* – why the number $(\frac{9803}{6931})^2$ is not equal to 2?", usually leads, with a bit of effort, to a proof that $\sqrt{2}$ is irrational. Is this effort worth it? Well I often ask that, and I suppose I feel in my heart of hearts that it isn't, but the alternative – simply telling one's students everything and having them promptly forget it – is just too awful to contemplate.]

Return to Report. *Up to that point I had deliberately not used the term 'irrational number'*, but now I did, and I gave them a potted history: that in early Greek Mathematics it was believed that all numbers were 'rational', and made some obvious points concerning the reasonableness of that belief; but that about the time of Pythagoras this belief fell apart. Their showing – the work of my group – that $\sqrt{2}$ was not a rational number, was a path that had been gone over many times by generations of mathematicians. I told them that a great mathematical hero of mine – G.H. Hardy – wrote in his 'A Mathematician's Apology' that he considered the Greek proof of the irrationality of $\sqrt{2}$ (together with Euclid's proof of the infinitude of primes) to be one of the finest pieces of classical Mathematics to have come down to us.

By way of giving them some idea that there were great unsolved questions in connections with $\sqrt{2}$, I told them that when I was a post–graduate student in London I attended a course by a very renowned mathematician – perhaps I should keep his identity a secret! – and that once, in conversation, some of us had asked him which question in Mathematics he would most love to settle; well, there were many, but if pressed to name just one of them it would be to know the decimal expansion of $\sqrt{2}$. I said that he didn't just mean being able to actually calculate its expansion to any degree of accuracy – a completely trivial problem which I discussed with them – but rather be able to find some non–trivial way of telling what the nth decimal place in the expansion of $\sqrt{2}$ would be.

Thus, while $\frac{1}{7}$ has decimal expansion .142857 (repeat), and so the n^{th} place of the decimal expansion of $\frac{1}{7}$ is completely determined by the remainder that n leaves on division by 6 (and I got correct answers to a few questions like: "what is the one hundred and twenty fourth place in the decimal expansion of one seventh?"), no such simple *deterministic* solution is known in the case of $\sqrt{2}$, nor for that matter for any (what I called) naturally occurring irrational number. {One can, of course, construct irrational numbers like 0.12345678910111213… which is easily seen to be irrational, and with a little effort can be seen (D. Champernowe, 1933) to be "normal in the base ten" (see [1]), and is not just irrational, but is in fact – as Kurt Mahler proved in 1937 – transcendental.}

I mentioned that whereas most people probable regard $\sqrt{2}$ as somehow being the first irrational number – almost certainly the first encountered in study – it could be said that a number called the 'golden ratio' was probably the first to actually be discovered. None had heard of this ration and so I spent some time describing how it arose; technically it boiled down to this: we sought two (positive) number *L* and *l*, with $L > l$, such that the ration of *L* to *l* was the same as *1* to $(L-l)$. (I asked them to try to find rectangles with integral sides which are 'almost golden', so ones with sides like 13 and 8 (with long to short side ration 13 to 8 (=1.625) which leads to the smaller rectangle with new ratio 8 to 5 (=1.6) – these two ratios are close in value), but not ones with sides like 13 and 7 (with long to short side ratio 13 to 7 (=1.857…) leading to the smaller rectangle with new ratio 7 to 6 (=1.16…) – producing two ratios that are not close in value. I choose not to mention the Fibonacci numbers.) That led to the equation:

$$L/l = l/(L-l), \text{ and cross-multiplying gave } L^2 - Ll = l^2,$$
$$\text{which tidied up to } L^2 - Ll - l^2 = 0.$$

Now, what to do with that? **RM** quickly observed that if you just took *l* to be one – and you could do that as it was the ration of *L* to 1 (and not *L* and *l*) that was of interest – then you just got the equation:

$$L^2 - L - 1 = 0$$

which he said was quadratic and could be solved in the usual way (the formula…). Here some of the younger members of the class were lost, and I just assured them that it wouldn't affect their understanding of what was to follow. (In the evening some of the older more experienced members of the class helped the younger ones on the classic solution:

$$\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Then $L^2 - L - 1 = 0$ has positive solution: 'golden ration' $= L = \frac{(\sqrt{5}+1)}{2}$.

It was immediately clear that if the latter number was rational then – setting $\frac{(\sqrt{5}+1)}{2} = \frac{a}{b}$ etc. – it would follow that $\sqrt{5}$ was also rational. We returned to that later.

Some of the younger ones had not heard of the 'Theorem of Pythagoras' concerning the sides of right–angled triangles, so I just spoke briefly about that and explained how it can immediately be used to calculate the hypotenuse of a right–angled triangle when the lengths of the other two sides are known. In particular, $\sqrt{2}$ can be thought of as being the length of the hypotenuse of the right–angled triangle whose two sides containing the right angle are both of unit length (and that $\sqrt{3}$ could be thought of as being the hypotenuse in the case where the other two sides have lengths 1 and 2 units, etc.). I told them also of the classic Greek geometric problem of 'duplicating the cube', from which question the cube root of two (a new notation for some of them) naturally arises. Now I will return to recording the class discussion.

JC: You see, there are so many numbers, and for each of them we have a question to ask: is it rational or irrational? You have settled for me the question concerning $\sqrt{2}$, but now what about, say $\sqrt{3}$; is it rational or not?

RC: It's also irrational. (**JC:** Can you give me proof?) Well, suppose it was rational. Then $\sqrt{3} = \frac{a}{b}$ for some $a, b \in$ Z, and so $b.\sqrt{3} = a$ and so $b^2.3 = a^2$. Then $3 \mid a$. (**JC:** Why?) Because 3 'has the property'. {**JC:** Good!. *Aside to reader*: My own regular students would argue (correctly) that a and b can't be even if $\frac{a}{b}$ is taken initially to be in 'reduced form', that one of them could not be even and the other odd (because 3 is odd), and so a and b must both be odd. But they are then not able to clinch a proof, which I leave to the reader}. Then $a = 3A$ for some $A \in \mathbf{Z}$ and so $b^2.3 = (3A)^2 = 9A^2$ and so $b^2 = 3A^2$ and so we also get $3|b$.

JC: Excellent! (I deliberately wanted to slow down a little for the benefit of the few in the group whom I felt were weak and so:) I take it that everyone follows that. Robert has just argued that if you have a rational number which is equal to the square root of 3 then the numerator and denominator can't be any old integers, they would have to both be divisible by three. You might like to use your calculators to find the approximate values of $\sqrt{3}$ and $\frac{70226}{40545}$.

There was then displayed:

$$\sqrt{3} = 1.732050808\ldots$$

$$\frac{70226}{40545} = 1.732050808\ldots$$

JC: You see the incredibly good agreement? But do we have equality? And if not between $\sqrt{3}$ and $\frac{70226}{40545}$, then maybe between $\sqrt{3}$ and $\frac{a}{b}$, for *some* integers a and b? Back to you Robert.

RC: You can't have such integers a and b, because if you did then they would both have to be divisible by 3, and all you have to do is to keep on dividing numerator and denominator by three until you eventually arrive at an impossibility.

I just spent a little time going over that, and wrote up the fine detail (as I consider it important that they acquire the skill – from example – of writing work up) and then continued. I will only briefly outline how the subsequent discussion went: $\sqrt{4} = 2$ is rational; $\sqrt{5}, \sqrt{6}$ and $\sqrt{7}$ are irrational – they (I picked on individuals to ask them to add the next bit of detail) were able to provide the details. With $\sqrt{8}$, one got to this point: "$b^2.8 = a^2$ so $8|a^2$ and… oh! You don't get $8|a$." Why not? "Because 8 'doesn't have the property'". Give me an example to show. "$8\nmid 4$, $8|4^2$". So what to do now? Someone said to just re–write $\sqrt{8}$ as two times $\sqrt{2}$ and it follows immediately that $\sqrt{8}$ is irrational from the fact that $\sqrt{2}$ is irrational. I asked who could think of some other numbers whose irrationality could be established in similar fashion, and I was instantly told: "root 12 (Why? "It's two root three"), root eighteen ("it's three root two"), root twenty ("it's two root five")… Then:

JC: Can you guess which natural numbers have rational square roots, and which ones have irrational square roots? Of course, just to make a correct guess is not the same thing as making a correct proof!

S: If a natural number is a square then it's square root is rational , but if a natural number isn't a square then its square root is irrational.

JC: That's correct, and if I had time I would love to show you how to prove it, but I want to move on… OK, you have guessed correctly as far as square roots of natural numbers are concerned, but I wonder if you can tell me what happens if you add two such numbers together, say, for example, root two plus root three?

JC: (after some silence): What would be a daft thing to say?

S: That root two plus root three is irrational because both root two and root three are irrational.

JC: Yes! That's right! That *really* would be a daft thing to say. It *is* true that if you add two rational numbers then you certainly get a rational number – that's completely trivial, you can easily prove it – but when you add two irrational numbers then *anything can happen*. Sometimes you get a rational number, but other times you get an irrational number. Can anyone give me an example of the former?

S: If you add root two and minus root two you get nought which is rational.

JC: Yes, that's true, but I personally would regard that as being just a little on the trivial side, as the same would be true of 'root three plus minus root three etc.' It's correct, it's correct, but can you give me an example not of this type?… OK, you can't, well think about it and let me know. In the meantime what about root two plus root three?

S: I think it's irrational.

JC: Let's hear the proof then.

If $\sqrt{2}+\sqrt{3}=\frac{a}{b}, a,b \in \mathbf{Z}$, then $(\sqrt{2}+\sqrt{3})^2 = \frac{a^2}{b^2}$.

Squaring out gives : $2+2.\sqrt{6}+3=\frac{a^2}{b^2}$, and then :

$2.\sqrt{6}=\frac{a^2}{b^2}-5$, and that's impossible as root six is irrational.

JC: That's right! It's that simple! … (pause) … but I'm a bit on the pedantic side and I'd like you to tidy up the last equation so as to isolate the root six and really be able to see the impossibility staring you in the face.

S: It just comes to: $\sqrt{6}=\frac{a^2-5b^2}{2b^2}$, which is rational, and which is impossible.

JC: Good; is impossible as we have already argued earlier that root six is an irrational number, and now a consequence of that is that root two plus root three is also irrational. One sees that kind of thing quite a lot in Mathematics – it's part of the beauty of it! – you establish some result, it may be small, it may be large, and as a bonus establish some other result which follows from it, and maybe another consequence from that… So, you have been able to handle root two plus root three; equally I'm sure you would be able to deal with, say, root two plus root five, root three plus root five, etc. What would you say – I wonder – about root two plus root eight?

S: It's irrational.

JC: Maybe it is, maybe it isn't. If you think it's irrational I wonder how you might prove it?

S: Just suppose that $\sqrt{2}+\sqrt{8}=\frac{a}{b}$, $a, b \in \mathbf{Z}$.

(**Aside**: I was greatly impressed that all of them who made contributions *now* but in the '$\in$ **Z**' when it was appropriate to do so – of course earlier in the week they hadn't been so careful; with my own regular students I have to continually pull them up on that. They will rattle off irrationality proofs without showing any sensitivity with regard to 'integers', and that happens in spite of the number of times that I say things like: "when one claims that root two irrational one is *not* saying that $\sqrt{2}=\frac{a}{b}$ is impossible for a and b being '*numbers*' – that would be clearly daft, since e.g. $\sqrt{2}=\frac{5.\sqrt{2}}{5}=\frac{\sqrt{6}}{\sqrt{3}}$, etc. – rather what one is claiming is that $\sqrt{2}=\frac{a}{b}$ is impossible for a

and b being 'numbers', but not any old kind of numbers, *it's for 'numbers' that happen to be integers*".)

S: continuation: Squaring gives $(\sqrt{2}+\sqrt{8})^2 = \frac{a^2}{b^2}$, and then you get

$$2+2.\sqrt{2}.\sqrt{8}+8=\frac{a^2}{b^2}, \text{ and so } 2.\sqrt{16}=\frac{a^2}{b^2}-10.$$

JC: (after a short silence): So, where does that get you? What are you going to say now? (and I wasn't going to say they should have left the 10 on the left hand side and on replacing the $2.\sqrt{16}$ by 8 obtain $18 = \frac{a^2}{b^2}$ which is impossible as 18 is irrational, as seen earlier.)

S: That doesn't look impossible.

JC: You're dead right; it certainly 'doesn't look impossible'. You are saying that – I take it – because both sides are rational?… So, what can you do now?

S: Ah! Don't bother squaring in the first place! Instead just say that if $\sqrt{2}+\sqrt{8}=\frac{a}{b}$ then $\sqrt{2}+2.\sqrt{2}=\frac{a}{b}$, so $3.\sqrt{2}=\frac{a}{b}$, etc.

JC: Good! You see the difference between the two approaches? The first – although it didn't have any errors in it – just didn't get us anywhere, but the second got us a result, a definite conclusion. (pause.) I'd like to look at some other kinds of numbers, this time cube roots rather than square roots. You remember I was telling you about the old Greek problem of the 'duplication of the cube' and how that led to having to think about the cube root of two? Now is that number rational or irrational?

RC: It's irrational. (**JC:** prove it.)
Suppose that $\sqrt[3]{2}=\frac{a}{b}$ for some $a,b \in \mathbf{Z}$, then $2b^3 = a^3$,
and so $2 \mid a^3$, and so $2 \mid a$.

JC: I'll accept that without quibble. Then?

RC: Then:
$a = 2A$ for some $A \in \mathbf{Z}$, and so $2b^3 = (2A)^3 = 8A^3$ and we get
$b^3 = 4A^3$, then $4|b^3$ and so $4|b$.

JC: (jumping on him immediately): No! no! no! That I do not accept; why don't I?

RM: Just because four divides the cube of an integer doesn't mean that four divides the integer. (**JC:** examples?) Four divides two cubed but four doesn't divide two, four divides six cubed but four doesn't divide six.

JC: Good. You see, Robert's argument was going along nicely; he was probably going to say something like 'four divides b, so two divides b, so two divides a

and b etc.' But his *attempted* proof came unstuck when he *tried* to claim that four had to divide b. What can be done, if anything at all?

RM: Don't use that $b^3 = 4A^3$ gives $4|b^3$, but instead just use that it gives $2|b^3$ and so then $2|b$, and use 'reduced form'.

So, we had a proof that the cube root of two is irrational; I went over it quickly and took up Robert's first observation that if $a \in \mathbb{Z}$ and $2|a^3$ then $2|a$. As that kind of observation is critical for these simple irrationality proofs I wanted to make sure that everyone possessed the requisite skill and so asked several questions like: "if a is an integer which leaves remainder 0 (then 1, 2, 3, 4) on division by 5, what least remainder does a^3 leave on division by 5?". Very quickly we saw that one could give irrationality proofs for $\sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \ldots$, and when I asked if anyone could guess the general result for cube roots of natural numbers I was immediately given the obvious answer. And the same for fourth, fifth, … roots.

I also spent some time asking them to investigate the rationality/irrationality of numbers like $\sqrt{1.5}, \sqrt{2.5}, \sqrt{3.5}$, *etc*., and very quickly we were into the question of general square roots of rationals, and so, for example, had to consider equations like $5a^2 = 7b^2$, where instead of having to consider the consequences of $5|b^2$ (or $7|a^2$) one has to consider the consequences of $5|7b^2$ (or $7|5a^2$). The bright ones were able to correctly argue, for example, that if $a \in Z$ and $7|5a^2$ then $7|a$. To bring the day to an end I went back to square roots of natural numbers and asked if anyone could argue for me that $\sqrt{93}$ is irrational. One person immediately (but too hastily!) said "suppose $\sqrt{93} = \frac{a}{b}$, etc., then $93|a^2$". Now a leaves ninety three possible remainders on division by 93, and so letting:

$$a = 93A + 0,1,2,3,\ldots,92 \text{ for some } A \in \mathbb{Z}, \text{ we get:}$$
$$a = 93A' + 0,1,4,9,\ldots,1 \text{ for some } A' \in \mathbb{Z}.$$

But it was immediately clear that we appeared to have a huge amount of crude computation to do to get the (correctly guessed) consequence, namely: $93|a^2$ implies 93|a. Someone then wisely suggested that what one should do was not to go from having $93b^2 = a^2$ to saying $93|a^2$ etc., but rather to notice that 93 is divisible by 3, and so a^2 is divisible by 3, and so a itself is divisible by 3. Then $93b^2 = a^2 = (3A)^2 = 9A^2$, for some $A \in \mathbb{Z}$, and so $31b^2 = 3A^2$, and so $3|31b^2$. But we had already seen that kind of situation before and so now could conclude that $3|b$. But then we have $3|a$ and $3|b$, and so we are once again into the business of commenting on reduced form. So the simple observation that 3 divides 93 really saved a lot of work! I suggested that they investigate other similar examples in their own time.

We were absolutely flying at that stage, but our day had come to an end.

Some written comments at the end of that day

- "I found proving that the $\sqrt{2}$ was an irrational number very interesting and logical. I understood most of what we covered today and found it very enjoyable." (a 14–year old)

- "I found the reason for finding integers that cannot be equal to $\sqrt{2}$ easy to understand." (a 15–year old) (**JC comment:** clearly some mis(lack of?) understanding there.)

- "Today was very interesting because I learned a lot about Greek mathematicians and history with relation to 'the golden ratio' and irrational numbers. Other than that I found the subjects of the day quite easy to understand." (a 15–year old)

- "I didn't really like trying to get the $\sqrt{2}$."

- "I found the golden ratio fascinating but I would love to know where those $\sqrt{2}$ approximations come from." (a 15–year old) (**JC comment:** This is pure joy for a teacher, having a student who wants to know something! Almost all of the following day's work was devoted to "those $\sqrt{2}$ approximations," and more …)

- "I found today's class extremely interesting, especially the golden ratio. I didn't quite catch on to the $\sqrt{2}$, but when I did, I kicked myself!" (a 14–year old)

- "I enjoyed learning about rational and irrational numbers. It was new but fairly easily understood. The class is still really interesting." (a 15–year old).

Friday 16th July 1993

I allowed the day to begin slowly by gently revising (for the benefit of the weaker ones) what we had done the previous two days. Although I was greatly pleased with what they had achieved – and I told them so – I felt nevertheless that I should let them know that the irrationality results that they had proved were at the easy end of the theory, and that there are certain number which are irrational, but whose proofs are *very much* more difficult, and that it would take *many years* of study to be able to follow them. I mentioned several such numbers, *e.g.*, π and $2^{\sqrt{2}}$.

I felt it would be of benefit to them if I was to say a few words about the *meaning* of the number $2^{\sqrt{2}}$ (and other such numbers), but without going into any great technical detail. First I discussed the meanings (with appropriate notation) of squares, cubes, fourth powers, etc., square roots, cube roots, fourth roots, etc., then the meanings of two–thirds powers (it wasn't all one way; I got them to do mental and calculator computation), and *rational* powers generally. Then, what is the meaning that mathematicians give to a number *like* two to the power of root two (which is an *irrational* power of two)? The *key* to it is simply this (I realise that I could have invited them to give me their ideas – I'm sure they would have had them! – but I wanted to move on quickly to studying those incredibly good rational approximations, especially as some of them had expressed an interest in that):

JC: I'll pick on two rational numbers, one *just* to the left of root two, the other *just* to the right of root two; let's say $\frac{7}{5}$ and $\frac{17}{12}$. Would you use your calculators to find the values of two to the power of seven over five and also two to the power of seventeen over twelve?

There was then displayed:
$2^{\frac{7}{5}} = 2.639015822\ldots$, and
$2^{\frac{17}{12}} = 2.669679708\ldots$

JC: Those two numbers are *quite close* together; but you can get two others which are *closer still*. You just replace that $\frac{7}{5}$ by a *slightly larger* rational number, but which is *less* than root two, and the other by a *slightly smaller* rational number, but which is *greater* than root two. You should do this sort of thing yourself in your own free time. The number 'two to the power of root two' is then *defined* to be – its *very meaning* is – that number which is at the *borderline* between those increasing numbers that come from replacing $\frac{7}{5}$ etc. and those *decreasing* numbers that come from replacing $\frac{17}{12}$ *etc.* … Mathematicians would say it was the common '*limit'* of those two collections of numbers. Basically that is how one defines '*a* to the power of *b*' in general, though you have to be careful when '*b*' is negative.

I told them about the famous list of twenty-three problems presented by the renowned mathematician David Hilbert at the International Congress of Mathematicians in Paris in 1900; that he had specifically mentioned the number $2^{\sqrt{2}}$ in the seventh of those, in which he had asked for a proof that a whole class of numbers – including $2^{\sqrt{2}}$ – are 'transcendental' (i.e., non–'algebraic'). I explained accurately but briefly what that meant, and said that it could be thought of as being a very extreme case of being irrational. I also told them that Hilbert's seventh problem was a generalisation of a conjecture that dated back to Euler – "my mathematical hero when I was a schoolboy" – and that it arose in the following way:

JC: If I take the two whole numbers eight and four, what power must I raise eight to, in order to get four?

All: Two over three!

JC: That's right, good, (and I wrote on the board: $8^{\frac{2}{3}} = 4$) and equally if I had asked what power four must be raised to, so as to give eight, you would have told me?

All: Three over two!

JC: (and I wrote $4^{\frac{3}{2}} = 8$) OK, that's easy. Suppose now I choose two and three; what power must I raise two to, so as to get three? (an inevitable silence) Of course you're stuck. That's because you're probable casting about trying powers like three over two, or four over three, or …, to see if they work. But they don't! 'They' being *rational* powers. There is *no* rational power of two which equals three! – (and we considered that quickly and saw why; it's very easy, of course, but it's real Mathematics) – but there is *some* power, *p*, of two which equals three (and wrote $2^p = 3$ for some *p*). Euler *felt* – he could not prove it – that that number *p*, and all *like* it, are 'transcendental', a term which he himself coined. The seventh problem of Hilbert was just a more general version of Euler's.

I also told them something of the subsequent history. Briefly: Hilbert's lecture in 1920 on the Riemann Hypothesis, Fermat's 'last theorem' (now Wiles' theorem! – Wiles had just made his announcement the previous month, and I had already told them about that) and the seventh problem of his. The great C.L. Siegel – in 1920 just a young student – was at that lecture, and, as reported in Constance Reid's biography of Hilbert **[2]**, Hilbert opined that he might live to see the first of these being settled, that maybe some of the young people present would live to witness the solution of the second, but as for the third? – Never!

But how wrong Hilbert was! By 1929 the young Russian mathematician A.O. Gelfond had settled a special case of Hilbert's seventh, the following year Kuzmin etc., and then in 1934 Gelfond, and, independently, Schneider (who was Siegel's student) settled the entire seventh problem. Also that in more recent times (the 60's) the great English mathematician Alan Baker had been able to prove the transcendence of – and thus irrationality of – numbers like $2^{\sqrt{3}} \times 3^{\sqrt{5}}$, and that for such (and other) revolutionary work he had been awarded a 'Fields medal' – which I said was the mathematical equivalent of the Nobel prize – at the International Congress of Mathematicians in 1970.

Then, after a short break, it was time to get down to work!

JC: Some of you have said that you would love to know how I got rational numbers that are very close to $\sqrt{2}$ etc. Well I'm not going to tell you how or where I got them from; instead you're going to find them yourselves. But just before we go looking for those rational numbers I'd like to show you something really silly!

(Aside: It was to have been our last day together and I thought a bit of fun was in order. I had first seen what I was about to do with them in **[3]**, which I had read when I was a student, and which I found delightful. At the time it had seemed like pure magic, and now I hoped they would feel the same.)

Yesterday when we found the value of the 'golden ratio' we saw it was the positive solution of $L^2 - L - 1 = 0$; we solved that equation by using 'the formula for solving quadratic equations', and we found L to be $\frac{(\sqrt{5}+1)}{2}$. That number is an irrational number. Why?

S: If $\frac{(\sqrt{5}+1)}{2} = \frac{a}{b}$ for some $a, b \in Z$ then $\frac{(\sqrt{5}+1)}{2} = \frac{2a}{b}$ and so $\sqrt{5} = \frac{2a}{b} - 1 = \frac{2a-b}{b}$, which is rational, but is impossible as we know that $\sqrt{5}$ is an irrational number.

JC: That's good. It's easy. The golden ration, $\frac{(\sqrt{5}+1)}{2}$, is an irrational number. It is not equal to the ratio of any two whole numbers. The *best* you could hope for would be to find pairs of whole numbers with ratios that are close to $\frac{(\sqrt{5}+1)}{2}$. Tell me what value your calculators give for $\frac{(\sqrt{5}+1)}{2}$.

There was then displayed: $\frac{(\sqrt{5}+1)}{2} = 1.618033989\ldots$

JC: We all know that that is just an approximation – admittedly a very good one. It couldn't but be an approximation since the left hand side is an irrational number and the right hand side, as it stands, terminating at that last '9', is rational. (pause). I would like to return to the equation $L^2 - L - 1 = 0$, which we have already solved using the 'formula', and make *another* approach at solving it: Let's re–write the equation as $L^2 = L + 1$ and divide both sides by *L*, producing $L = 1 + \frac{1}{L}$. Nothing much had happened. (pause). But just suppose we *replaced* that *last L* (the one in '$\frac{1}{L}$') with the *initial* value of *L*, namely that *L* is $(1 + \frac{1}{L})$, we would have:

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{L,}}$$

and if we *repeated* what we have just done, namely replace that last *L* with what we initially knew *L* to be – or we could if we wished replace it with what we *now* know *L* to be, it would just lead to jumping on a step – we would now have:

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{L}}}$$

and we could go on and on doing that sort of thing for miles and miles of blackboard space apparently getting nowhere.

On the blackboard I wrote several iterations until I had ten numerators/denominator bar lines:

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + \cfrac{1}{1 + \cfrac{1}{L}.}}}} \qquad \textbf{...(E)}$$

Before I came into class I was intending saying what I now went on to say, but I was much aided by a wonderful piece of good fortune: at the end of the previous day I had left certain calculations on the board, but before we began our work on Friday morning one of the cleaning staff had wiped everything off the board, and I now asked them to imagine that the above equation **(E)** had been on the board when we finished on Thursday but that overnight one of the cleaning staff had come into our room and had erased nothing but that *final L* in **(E)**. What would we now see on the board? Simply this.

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + \cfrac{1}{1+1}}}}. \qquad \textbf{...(E')}$$

Then:

JC: This equation, **(E')**, now has *only one L* in it, and gives an *actual value* for *L*. Let's just work together and find what that value is!

Of course I took the initiative and *forced* the pace (anyone who teaches will recognise that this is one of those occasions when this is what one should do, otherwise momentum is lost and things can fall flat. And anyway, magic is magic and a magician rushes you):

JC: What is one plus one (pointing to '$1+1$' in **E'**)?

All: Two! (And I replaced the '$1+1$' with 2)

JC: And then one over two is just one over two, and one *plus* one over two is just three over two (and I now altered **E'** so that the tail–end of it looked like this:)

$$\ddots + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}}$$

Now what is one over three over two.

All: Two over three! (I made the corresponding changes in **E'**)

JC: And one plus two over three? **All:** Five over three!

JC: And one over five over three? **All:** Three over five!

JC: And one plus three over five? **All:** Eight over five!

JC: Good! You've got the idea! We'll keep on doing this together until we get back to the start and see what crazy value it throws up for us for *L*! (Aside: I would ask a reader who is not familiar with the 'theory of continued fractions' to perform the subsequent calculations. And finally:)

All: One hundred and forty four over eighty nine!

It all seemed such harmless fun!

JC: All right. We knew how to find the exact solution of the equation $L^2 - L - 1 = 0$; we found it (the positive solution) to be $\frac{(\sqrt{5}+1)}{2}$, and we knew that it is an irrational number. Then we did something crazy, really crazy, and it gave us a number. Not any old kind of number. A *rational* number. One hundred and forty four over eighty nine. (pause). Oh, by the way, you might like to us your calculators to find the value of one hundred and forty four over eighty nine.

Directly underneath the already displayed:

$$\tfrac{(\sqrt{5}+1)}{2} = \underline{1.6180}33989\ldots, \text{ I now displayed their:}$$
$$\tfrac{144}{89} = \underline{1.6179}77528\ldots$$

I couldn't be certain how many of them were amazed at this – I was so excited myself – but certainly some of them were. (One later wrote: "Friday's class was the class I most enjoyed. I found everything interesting and with a little thought I understood it all. I found the continued fraction expansions particularly fascinating…")

JC: Isn't that just wonderful? We did something that *appeared* to be daft, but it turned out to be not so daft after all. And that was by wiping out that last *L* in our **E** which had whatever number of terms in it. What do you think we would get if we had many more terms and then did the same kind of thing that we've just done?

S: You would get a rational number which was closer than the previous one.

JC: Yes! Yes! And if you took more and more terms you would continue to get closer and closer still! But it all happens in an *incredibly organised way*. You *alternately* get rational numbers which are *smaller* and *greater* than $\frac{(\sqrt{5}+1)}{2}$. What we are playing with here are what mathematicians call 'continued fractions', 'continued' because they go on and on. They don't go on and on for all numbers. But for *all irrational numbers* they do. For rational numbers they stop, they terminate. You might like to *systematically* delete the *L* on the right hand side in equations like **E** just to see the rational numbers that you get – they are called the 'convergents' of $\frac{(\sqrt{5}+1)}{2}$ and their 'limit' is $\frac{(\sqrt{5}+1)}{2}$ – and then compare their values with $\frac{(\sqrt{5}+1)}{2}$. Let's do the first several…

If I had had more time with them I would have loved to have introduced them to more work on continued fractions (I just returned briefly to them later), but it might have been rather one way and I wanted to return to work where I hoped they would (and did) make fast progress themselves.

JC: I'd like to return to the problem of finding really good rational approximations to numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots$ But first I hope we can agree as to what we mean by "really good". I'd like to make a definition of "really good" which is sensitive to the *quality* of approximations. With $\sqrt{2}$, for example, we say that the rational number $\frac{17}{12}$ when squared gave $\frac{289}{144}$. The latter isn't two – we could never get two in such a manner, we all know that at this stage – but it is *very* close to two. *Had* that numerator '289' been just smaller by one then we would have had $\frac{288}{144}$, which is two exactly.

And the same sort of thing if we squared, say $\frac{41}{29}$. Here we get $\frac{1681}{841}$, which is very close to two. So close to two that a *mere* increase of one in the numerator 1681 would give $\frac{1682}{841}$, which is two exactly. But if we squared the rational number $\frac{10}{7}$ we would get $\frac{100}{49}$. That is also decently close to two, but instead of having to reduce the numerator by just one to get the number two we would have to reduce it by two.

What we really want in considering the irrational number $\sqrt{2}$ is to find *lots* of natural number pairs (p, q) like (17, 12) and (41, 29) such that:

either $(\frac{p}{q})^2 = \frac{p^2}{q^2} = \frac{(2q^2-1)}{q^2}$, i.e. $\underline{p^2 = 2q^2 - 1}$,

where an increase of one in the numerator would produce the number two exactly,

or $(\frac{p}{q})^2 = \frac{p^2}{q^2} = \frac{(2q^2+1)}{q^2}$, i.e. $\underline{p^2 = 2q^2 + 1}$,

where a decrease of one in the numerator would produce the number two exactly.

Let's agree to call $\frac{p}{q}$ an **L–approximation to $\sqrt{2}$** when the former happens, and to call it an **R–approximation to** $\sqrt{2}$ when the latter happens. L for 'left' and R for 'right', because in the first case $\frac{p}{q}$ will be just to the left of $\sqrt{2}$, and in the second case just to the right of $\sqrt{2}$.

Well, off you go then! You might like to work in pairs, calculating and noting. Checking systematically increasing values of q – the denominator – I'll start us off by calling out the value of q, you then call out the values of $(2q^2 - 1)$ and $(2q^2 + 1)$ and also tell me if these number are squares or not. So, $q = 1$?

All: 1 and 3, 1 squared and not a square.

JC: So $q = 1$ gives us an L–approximation to $\sqrt{2}$, namely $\frac{1}{1}$, but no R–approximation to $\sqrt{2}$. Now, $q = 2$?

All: 7 and 9, not a square and 3 squared.

JC: So $q = 2$ gives us no L–approximation to $\sqrt{2}$, but does give us an R–approximation to $\sqrt{2}$, namely, $\frac{3}{2}$. And $q = 3$?

All: 17 and 19, not a square and not a square.

JC: $q = 4$?

All: 31 and 33, not a square and not a square.

JC: $q = 5$?

All: 49 and 51, 7 squared and not a square!

JC: So $q = 5$ produces an L - approximation to $\sqrt{2}$, namely $\frac{7}{5}$ and no R–approximation to $\sqrt{2}$. OK, you're on your own. Calculate! How far? Let's say down as far as twenty. Whatever you find tell me whether they're L– or R–approximations.

All: (after a minute): There's only one more down to 20, it's from $q = 12$, producing the R–approximation $\frac{17}{12}$.

On the board were written:

$$\tfrac{1}{1}\text{(L-)} \qquad \tfrac{3}{2}\text{(R-)} \qquad \tfrac{7}{5}\text{(L-)} \qquad \tfrac{17}{12}\text{(L-)}$$

RM: (in an instant!): I think I see a connection.

JC: (staying calm and giving nothing away): What connection?

RM: If you add the one and the one you get two, if you add the one and the two you get three, if you add the three and the two you get five, if you add the two and the five you get seven, if you…

JC: (wishing that he would give up 'cookbook' Mathematics and make a precise general statement): So what would be another approximation after the one coming from $q = 12$?

RM: Forty–one over twenty–nine.

JC: And what kind of an approximation might that be?

RM: An R–. They seem to go left, right, left, right, left,…

JC: Well let's check it and see. Is forty–one over twenty–nine an R–approximation to root two?

S: (after some seconds): Yes.

JC: And what about after that one?

S: (after some seconds): You'd get ninety–nine over seventy.

JC: And that's an L–approximation. And then you'd get an R–, and then you'd get an L–… I'll come back to that in a moment, but first I'd like to ask another question. We went looking for L– and R– approximations to $\sqrt{2}$, and in no time at all we have found some. But we didn't just find some, Richard *thinks* he has found a connection, has maybe found some way of finding more of them from the few already found. He could be right, but he could be wrong; who knows? Richard's 'connection' threw up *q* equals twenty–nine after the case *q* equals twelve, and then threw up *q* equals seventy after *q* equals twenty–nine. But all of that leaves one obvious question to be asked – well, obvious to my way of thinking. What do you think I have in mind?

RC: Is *q* equals twenty–nine the next *q* after twelve, is *q* equals seventy the next *q* after *q* equals twenty–nine.

JC: Ah! Excellent. Music to my ears. I think I know what you mean, but would you just be a bit more precise as to what you mean by "next"?

RC: I mean are there any values of *q between* twelve and twenty–nine that might give you and L– or an R–, any between twenty–nine and seventy that might give an L– or an R–?

JC: Wonderful. You see, the question is this: Richard saw a pattern – or thinks he has seen a pattern – it certainly seems to be successful, so far anyway!, but have some other L–'s or R–'s been missed out, jumped over? Well let's check and see. We have to check *q* equals thirteen, fourteen, and so on down to *q* equals twenty–eight. By chance that comes to sixteen tests to be done and there are sixteen of you, so there is one each to check. Off you go (and I just allocated these numbers, one to one).

All: (in seconds): There are no missing L–'s or R–'s between twelve and twenty–nine.

I didn't check on individuals, and it might have been possible that some didn't actually do what they should have done. But because it was so simple I didn't wish to embarrass anyone by putting them on the spot, and continued:

JC: That's right. And what do you suspect happens between twenty–nine and seventy?

All: There are no missing ones in there either.

JC: Yes, you've guessed correctly. But there's a *lot more* checking to do for that one. You've got to test q equals thirty, thirty–one, thirty–two, and so on down to sixty–nine. In your own time you should pick at random on a number in that range and check it to verify that it does not give you an L– or an R–approximation. Let's just do one together. Someone please pick on a value.

A value was offered and we tested it together. Then:

JC: I'll say something more about that later, but now how about Richard's recipe for producing more and more L–'s and R–'s? How can we prove that it's not just a fluke, but that it does actually work, not just for the couple of cases that have sprung up, but that it *continues* to work? So, what *general* claim are we making, and how can we prove it? (There was a lull and nobody was saying anything, not even Richard.) Maybe I'm not making myself clear enough, so how about this? Please take a page, put your name on it, and on it complete the following sentence: if a over b is an L–approximation to root two, then *what* over *what* is an R–approximation to root two?

(I looked at them later, and report on them now. Three of them seemed to be lost [writing: "when $\frac{a}{b}$ is an L - approx. then $\frac{b}{a}$ is an R - approx.", "when a is an l approx. $a+2b$", and something crossed out], but the other thirteen clearly understood, though the standard of presentation varied greatly, ranging from the majority [three of whom had only one year of secondary schooling] straightforwardly excellent ["when $\frac{a}{b}$ is an L–approximation then $\frac{(a+2b)}{(a+b)}$ is an R–approximation"], through some inelegant expressions [e.g. "when $\frac{a}{b}$ is an L - approximation then, $a+b=c$ where c is the denominator and $b+c=d$ where d is the numerator. Then $\frac{d}{c}$ is an R–approximation"], to the [one only] very badly expressed, which was full of horrors ["when $\frac{a}{b}$ is an L app., then X is an R app., $X=a+b=\frac{(a+b)}{b}=\frac{(a+b+b)}{(a+b)},=\frac{(a+2b)}{(a+b)}$"].)

Then:

JC: So who can complete the sentence for me?

S: When a over b is an L–approximation then a plus two b over a plus b is an R–approximation.

JC: Don't forget the bit about the root two. When a over b is an L–approximation to root two then a plus two b over a plus b is an R–approximation to root two. OK, how to prove this though?

No one, not even the best of them, had any idea how to go about it. So I had to give a shove:

JC: Ask yourself: what does it mean to say that *a* over *b* is an L–approximation to root two? It means that *a* and *b* are integers – and you can take them to be positive, in practice that's what they'll be – such that *a* squared minus two times *b* squared is equal to minus one. And what does it mean to say that *a* plus two *b* over *a* plus *b* is an R–approximation to root two? It just means that *a* and *b* are integers such that *a* plus two *b* all squared minus two times *a* plus *b* all squared comes to one, plus one that is, and not as with the other one where it is minus one.

On the board was written:

$$\tfrac{a}{b}\text{, an L}-\text{app. to }\sqrt{2}\ \underline{\text{means}}: a,b \in \text{N and } a^2 - 2b^2 = -1$$
$$\tfrac{(a+2b)}{(a+b)}\text{, an R}-\text{app. to }\sqrt{2}\ \underline{\text{means}}: (a+2b)^2 - 2(a+b)^2 = 1$$

JC: The problem now is this: if you are given that the first of these equations holds, how can you show that the second of them must also hold?

I will spare the reader the details (one of them wanted to prove the second held by the usual horror of supposing that it held, and then to show that it held…), but I had to intervene and essentially do it myself, and I suppose that now they actually learned something from me, a small skill, and now was written on the board.

Simple Theorem: If $a,b \in \text{N}$ and $\frac{\text{a}}{\text{b}}$ is an L – approximation to $\sqrt{2}$ then $\frac{(a+2b)}{(a+b)}$ is an R - approximation to $\sqrt{2}$.

Proof: $\underline{\text{Since}}\ \frac{a}{b}$ is an L – approximation to $\sqrt{2}$ $\underline{\text{then}}$ $a^2 - 2b^2 = -1$.

$$\text{Now } (a+2b)^2 - 2(a+b)^2 = (a^2 + 4ab + 4b^2) - 2(a^2 - 2ab + b^2)$$
$$= a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2$$
$$= -a^2 + 2b^2 = -(a^2 - 2b^2) = -(-1) = 1,$$

because we already know that $a^2 - 2b^2 = -1$.

So $\frac{a+2b}{a+b}$ is an R – approximation to $\sqrt{2}$.

They could do the elementary algebraic part of this; that is they could tell me that "*a* plus two *b* squared equals *a* squared plus four *ab* etc.", but making the above start of proof was not something that occurred to them. However, having once seen what to do, there was no stopping them.

JC: But going back to Richard's observation that the approximations seem to go left, right, left, right, …, what should we next think about (Aside to reader: I agree, it's a leading question, but I wanted to give them a chance to get back to something like their usual selves)?

S: What about if *a* over *b* is an R–approximation?

JC: Of course. We went looking for L–'s and R–'s for root two. We found a few. Richard thought he saw something. He was absolutely right. We made a guess about how to form an R–approximation to root two from an L–approximation to root two, and we verified an *actual* numerical case of it.

We might have verified hundreds, thousands of such cases, but that would not constitute a proof of the general assertion. Then we were able to find a proof of it, which with a bit of hindsight we see – yes? – was easy. Now we ask if something similar can be done if we have an R–approximation. And what do you think, what can we say?

S: You get the same rule. (**JC:** namely?) If *a* over *b* is an R–approximation to root two then *a* plus two *b* over *a* plus *b* is an L–approximation to root two.

Of course they could prove it. They saw that it was just a matter of going through the proof that we already had and making obvious sign changes. Then (a big surprise in store for them!), but I gave nothing away:

JC: Well, then off you go, working in pairs, and tell me what you find when you do the same sort of thing with root three.

S: How far should we try?

JC: Well how about as far as you did earlier, down to twenty?
After some minutes they finished their calculations and all had correctly found that root three has no L–approximations and three R–approximations with denominator ranging from one to twenty, the latter being (displayed on board):

$$\tfrac{2}{1}\,(2^2 - 3.1^2 = 1), \qquad \tfrac{7}{4}\,(7^2 - 3.4^2 = 1), \qquad \tfrac{26}{15}\,(26^2 - 3.15^5 = 1)$$

Many of them seemed genuinely perplexed that no L–approximations had cropped up; they felt there *should* have been some. One of them even suggested that what was going to happen was that if we checked on a little further we would come across three L–approximations on the trot (followed by three more R–approximations, …). We collectively tested some more values of q (21, 22, 23, …) and found no L's or R's. Then:

RC: I think I see a pattern for the R–approximations. If you add the numerator to *twice* the denominator you get the next denominator. With the two over one, that gives you two plus two times one, which is four, which is the second denominator. Then with the seven over four, that gives you seven plus two times four; that's fifteen, which is the third denominator. And if you then took the twenty–six over fifteen and added the twenty–six to two times fifteen it would give you the next denominator. (**JC:** which is?) Fifty–six.

JC: (keeping calm): Does that work? What do we have to do to see if that is correct?

S: Work out fifty–six squared, multiply that by three, add one, and see if the number you then have is a square.

JC: Let's do it. You get what?

All: Three thousand one hundred and thirty–six, nine thousand four hundred and eight, nine thousand four hundred and nine, and the square root of that comes to ninety seven!

JC: (letting go): We must be on to something here! This is just great! Come on though, give me a precise mathematical statement of what we might be on to here.

S: If *p* over *q* is an R–approximation to root three, then *p* plus two *q* is the denominator of the next R–approximation to root three.

JC: Good, good, but be careful. You have used the term "next", but it mightn't be "next" at all, it might just be "another". And maybe that gives us the next or otherwise denominator, but what about the corresponding numerator?

On the board was:

$$\frac{2}{1}, \frac{7}{4}, \frac{26}{15}, \frac{97}{56}. \text{ R – approximations to } \sqrt{3}.$$
$$\frac{p}{q} \text{ an R - approximation to } \sqrt{3}. \ \frac{?}{(p+q)} \text{ an R – approximation to } \sqrt{3}\,?$$

Several equivalent suggestions were simultaneously called out: "the difference between the numerator and denominator is the same as the sum of the previous numerator and denominator" $(7-4=2+1,\ 26-15=7+4,\ 97-56=\ldots)$ "if you add the old numerator and denominator to the new denominator, you will get the new numerator" $(2+1+4=7,\ 7+4+15=26,\ldots)$

JC: So with *p* and *q* being the numerator and denominator of the old R–approximation to root three, and maybe – because remember we haven't proved it – *p* plus two *q* being the denominator of the next, or maybe just another R–approximation to root three, what are you telling me – and I want to hear it in terms of *p*'s and *q*'s – might the numerator of the new R–approximation to root three?

S: Two *p* plus three *q*.

JC: That is certainly what it would have to be if what you have told me is correct. "The different etc." would give that if the new numerator was *X*, then *X* minus *p* plus two *q* equal to *p* plus *q* would give that *X* equals two *p* plus three *q*, and "if you add etc." would give that *X* equals *p* plus *q* plus *p* plus two *q*, namely two *p* plus three *q*. So we have the following conjecture or guess:

$$\text{If } \frac{p}{q} \text{ is an R - approximation to } \sqrt{3} \text{ then } \frac{2p+3q}{p+2q} \text{ is an R - approximation}$$
$$\text{(it might be the next one - that's another issue) to } \sqrt{3}.$$

Who can prove this, if it is true?

The following proof just wrote itself; I only wrote up what I was told.

Proof : Since $\frac{p}{q}$ is an R - approximation to $\sqrt{3}$ then $p^2 - 3q^2 = 1$.

(To see if $\frac{(2p+3q)}{(p+2q)}$ is an R - approximation to $\sqrt{3}$ we need to

look at the value of $(2p+3q)^2 - 3(p+2q)^2$, to see if it comes to 1 or not).

Now : $(2p+3q)^2 - 3(p+2q)^2 = 4p^2 + 12pq + 9q^2 - 3(p^2 + 4pq + 4q^2)$

$= 4p^2 + 12pq + 9q^2 - 3p^2 - 12pq - 12q^2$

$= p^2 - 3q^2 = 1$ (horray!)

and so $\frac{(2p+3q)}{(p+2q)}$ is indeed an R - approximation to $\sqrt{3}$.

JC: It's very easy really, isn't it? The thing was, of course, to have exercised judgement, to have had your eyes open, to have noticed things, to be guided by what you already knew. To have made the correct choice – that '$\frac{(2p+3q)}{(p+2q)}$' didn't lead to a successful conclusion. (pause). OK, we've found a way of producing an R–approximation to root three from an R–approximation to root three, but what about the question of L–approximations to root three. Are there any? If there are, then are they connected in any nice structured way as we've just seen with the R– ones?

There was silence which I allowed to last for just a while, and then:

JC: May I just point out a little something? If you do find an L–approximation to root three then the above means of producing an R–approximation to root three, from one already possessed, will also produce an L–approximation to root three from the one that has been found.

We all just checked that the above proof immediately altered to produce the conclusion:

$$(2p+3q)^2 - 3(p+2q)^2 = -1 \text{ if one started with } p^2 - 3q^2 = -1.$$

JC: So we are still stuck with the unresolved question: can we find an L–approximation to root three?

RM: (a little tentatively): I don't think root three has any L–approximations.

JC: Can you prove it? Let's hear your reasons.

RM: Well if there were any then you would have $p^2 = 3q^2 - 1$ for some integers p and q.

JC: So you would. That's precisely what's *meant* be root three having an L–approximation in the first place. And?

RM: But when you divide *p* by three there are only three possible remainders – nought, one or two – and from those you get that *p* squared leaves remainders nought, one or *one*; but then p squared plus one leaves remainders one, two or two. But the equation *p* squared equals three *q* squared minus one would give that *p* squared plus one was equal to three *q* squared which leaves remainder nought on division by three.

JC: Exactly! That's it! Root three just hasn't got *any* L–approximations.

I went over his proof and wrote it out in notional detail. We were now getting close to the end of our time together and so very quickly I drew their attention to certain obvious question:

JC: Now look at all the questions we have facing us. What about root five, root six, root seven, …? Do they have L– and R– approximations? Maybe some behave like root two and have both kinds. But even if they do, are there the kinds of connections between them that we've seen? Maybe some behave like root three, where there were R–'s but no L–'s? Others may have L–'s but no R–'s? And some may have no L–'s or R–'s. In your own free time you might like to investigate these questions and see what you come up with.

And that's what we get into by asking about *square* roots (for which we had been using the common shorthand 'root'), but you and I know that it's not just square roots that give us lots and lots of irrational numbers. There are the cube roots, fourth roots, fifth roots, … So, what about those? We can ask the obvious analogous questions. Say, for example, we start with the cube root of two, which we know is irrational – *meaning* that there are no integers *p* and *q* such that *p* over *q* cubed is equal to two. Well, as with square roots, we can ask for the next best thing (written up:)

$$\text{Are there } p, q \in \mathrm{N} \text{ with } (\tfrac{p}{q})^3 = 2 - \tfrac{1}{q^3} \text{ or } (\tfrac{p}{q})^3 = 2 + \tfrac{1}{q^3}\,?$$

$$\text{i.e. are there } p, q \in \mathrm{N} \text{ with } p^3 = 2q^3 - 1 \text{ or } p^3 = 2q^3 + 1?$$

We could call these 'L– and R–approximations to the cube root of two'. And you could investigate L– and R–approximations to the cube roots of four, five, six, seven, … You might find some, and you ask if there are connections between them… By the way, just stick your necks out and tell me what you think happens – I know we haven't actually done any work to back up any guess we might make, but just have a go anyway.

All: You would just get the same sort of thing happening as already happened with square roots!

JC: Well it wasn't a fair question of me to put to you, and you've just given me the natural response. But you are in for an incredible surprise. In moving up from square roots to cube roots we have entered into a *quite different world* altogether! Although it is no more difficult to prove that the cube root of two is irrational that it is to prove that the square root of two is irrational, in considering their respective L– and R–approximations the outcome is so radically different.

There is a beautiful theorem due to Delauney–Nagell (see e.g. **[4]** or **[5]**) – the first a Russian, the second a Norwegian – which reveals that if you choose any natural number *d*, then it's cube root has either no L–approximations – that is not a surprise to us, as we've already seen that sort of behaviour already – *or else only ONE L–approximation*! The same for its R–approximations, and similar results for fourth roots, fifth roots, …

Our time was now up…

I had been asked to work with them for the first week only, but the following day I spoke with Alastair Wood and begged him to allow me an extra day with them – if it didn't interfere with his plans and those of Fiona Lawless – and he granted my wish. Since this report is already quite long, and the reader will have a good idea as to how we worked, I do not intend giving a similarly detailed account of that last day. I would estimate that in the last day we did as much work as in any two days of the first week, but I would like to record just a sketch of the content of that day's work.

Day six, Wednesday 28th July 1993

My intention was to continue from where we had left off. I was especially keen to reveal to them the secret of the laws for the connections between the L– and R–approximations. What do I mean by that 'secret'?

Consider the task, for example, of trying to find a connection between the L– and R–approximations to $\sqrt{19}$. Imagine trying to do that, *using the approach with which they were already familiar*. Now that approach (wonderful and all though it was, and they had learned a great deal about it) would be *completely hopeless*.

$\sqrt{19}$ has no L–approximation (of course one wouldn't have been so foolish as to go blindly looking for any of them when [being alert to the possibility that there *might not* be any] a moment's reflection enables one to prove that there *aren't* any) but it does have R–approximations. The first of those is $\frac{170}{39}$, and would have been found fairly quickly. But the second one is $\frac{57799}{13260}$, and in trying to find it one might well have jumped to the too hasty conclusion that there wasn't a second one.

But even if one had had the patience to have persevered and eventually found it, one would need the divine powers of an Euler or a Ramanujan to see the connection between them. (The third R–approximation to $\sqrt{19}$ is $\frac{19651490}{4508361}$. The fourth, fifth, sixth are … . If you are not familiar with this corner of Number Theory and are wondering how this is done, all will be revealed later.)

Just before I was about to start I was informed that we were to have an observational visit by DCU's President (Dr. Daniel O'Hare – the Talented Youth programme at DCU is his brainchild), Alastair Wood, and some others. I made an on-the-spot decision that rather than start with the continuation of our L– and R–work, and be so far into it that our visitors would not know what we were up to when they arrived, I would instead begin with something alternative and completely unrelated, take a break when our visitors arrived, and then – with some revision of what we had already done in the first week – launch into our L– and R– work.

Brief summary of topics discussed before the arrival of visitors

- Euclid's theorem on perfect numbers: If $n \in \mathbf{N}$ and $(2^n - 1)$ is prime, then $2^{n-1}.(2^n - 1)$ is perfect.

- For which $n \in \mathrm{N}$ is $(2^n - 1)$ prime? Numerical experimentation quickly suggested $(2^n - 1)$ is composite when *n* is composite (and I showed how to prove that) but that it is a real problem – in fact it is one of the greatest unsolved questions in Mathematics! – as to what is the complete picture when *n* is prime.

- The 'Mersenne numbers' $M_p = (2^p - 1)$, *p* prime. Which values of p make M_p be prime and which make it composite?

  M_p is prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, \ \ldots$
  M_p is composite for $p = 11, 23, 29, 37, 41, 43, 47, 53, 59, \ \ldots$

- But how can one test primality of these numbers? There is the crude ancient test of Eratosthenes, which is quickly seen to be completely useless once *p* starts to get large.

- Euler's theorem on prime divisors of the Mersenne numbers (the first significant advance in testing the primality of M_p):

  If p is an odd prime then any prime *q* that divides M_p,
  must leave remainder 1 on division by 2*p*?

- Combining the latter with Eratosthenes allowed us to test the primality of, *e.g.* M_{17}, because any prime divisor of M_{17} (whose square root is between 362 and 363) must leave remainder 1 on division by 34, and so one *only* needs to test possible prime divisors from the list: 103, 137, 239, 307. None of these divide M_{17}, and so it follows immediately that M_{17} is prime.

- The Fermat numbers, with which they were already acquainted from the first week. How to test primality? The Eratosthenes test is once more completely useless after the first five of them (3, 5, 17, 257, 65537 – just the ones that Fermat knew were prime)

- Euler's theorem on prime divisors of the Fermat numbers (the first significant advance in testing the primality of F_n):

If $n \in \mathbf{N}$ then any prime dividing F_n leaves remainder 1 on division by 2 to the power of $(n+1)$.

- Combining this Euler result with the Eratosthenes allowed us to test F_4 (= 65537) and F_5 (= 4294967297). The first of these has square root just larger than 256 and so we *only* needed to test possible prime divisors from the list: 97, 193. Checking that neither of these divided F_4 showed F_4 is prime. Then we looked at F_5, and as its square root is just larger than 65536, it meant we only had to test for possible prime divisors from the list: 193, 257, 449, 577, 641, … (ending with the largest prime less than 65536 which leaves remainder 1 on division by 64). We checked each of 193, 257, 449, 577 in turn, found that none of them divided 4294967297, but on testing 641 we found – as Euler himself did (they already knew from the first week that $641|F_5$, but only as a *fact*. Now they saw where that came from) – that F_5 is composite.

Had I had more time I would have introduced them to the remarkable Lucas–Lehmer test for the Mersenne numbers, and to the equally remarkable Lucas–Pépin test for the Fermat numbers, but our visitors arrived and I spoke with them while my students took a short break.

After the break

- A quick revision of what we already knew about L– and R–.

- I asked them to investigate L– and R–approximations to $\sqrt{5}$. When they had only found $\frac{2}{1}$(L-) and $\frac{9}{4}$(R-) several called out "oh! It's *p* plus two *q*!" (it just jumps out at you). But is it right? It would predict 17 for the next L–denominator. And what does $(5.17^2 - 1)$ come to? To 1444, which is 38^2, giving $\frac{38}{17}$ (L–)! So they knew they *must* be on to something!

Frantic scramble to find the predictor for the numerator. A few too hasty suggestions tried and discarded before "two *p* plus five *q*" was tried – it gives the '9' and '38' of course, and the *double* prediction of the R–approximation:

$$\frac{(2 \text{ times } 38+5 \text{ times } 17)}{(38+2 \text{ times } 17)}, \text{ namely } \frac{161}{72}, \ (161^2 - 5.72^2 = 1)$$

clinches it! But does it? I asked them to write out a formal proof of:

"if $p, q \in \mathrm{N}$ and $\frac{p}{q}$ is an L– (R–) approximation
to $\sqrt{5}$ then $\frac{2p+5q}{p+2q}$ is an R – (L–) approximation to $\sqrt{5}$."

I collected their work and will just report that eleven of them gave perfectly acceptable
proofs (including one 14–year old girl, **Sharon Murray**, two years of secondary schooling, who had moved up from the other group *after* the first week), but a common error in the work of the other six was to write:

$$\text{“}(2p+5q)^2 = 4p^2+25q^2\text{, etc.”}$$

What about L– and R– for $\sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \ldots$? Someone argued correctly that $\sqrt{6}$ has no L–approximations, but by the *longish* route of considering the *six* possible remainders that ‘*p*’ leaves on division by 6, but that was immediately improved upon by: “if $\frac{p}{q}$ is an L– app. to $\sqrt{6}$ then we have $p^2 = 6q^2 - 1$, and $p^2+1 = 6q^2 = 3(2q^2)$ is impossible since we know (earlier work) that $3 \nmid (p^2+1)$. This is, of course, just another example of a saving device already encountered in connection with irrationality proofs.

- Which other number could be argued in similar fashion?

These: $\sqrt{12}, \sqrt{15}, \sqrt{18}, \sqrt{21}, \sqrt{24}, \sqrt{27}, \sqrt{30}, \ldots$
But what about: $\sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{13}, \sqrt{14}, \sqrt{17}, \ldots$?

It arose *naturally*, and obviously, that square roots of *primes* cause the most work.

Of the primes, which ones don't have L–'s?, but which do?:

Don't (we collectively argued): $\sqrt{3}, \sqrt{7}, \sqrt{11}, \sqrt{19}, \ldots$

Do: $\sqrt{5}, \sqrt{13}, \sqrt{17}, \ldots$ found by trial, and we saw that if you *tried* to argue that there were none, the ‘proof’ falls flat. For example, keeping an open mind about $\sqrt{13}$, if you try to prove that it has no L–approximations, by examining the remainders that (p^2+1) leaves on division by 13 you get:

For: $p \in \mathrm{Z}$, $p = 13P + 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$, some $P \in \mathrm{Z}$.
Then: $p^2+1 = 13A + 1,\ 2,\ 5,\ 10,\ 4,\ \underline{0},\ [11,\ 11,\ \underline{0},\ 4,\ 10,\ 5,\ 2]$, $A \in \mathrm{Z}$.

Because of that ‘*0*’ it *doesn't* follow that $13 \nmid (p^2+1)$ for <u>all</u> $p \in \mathrm{Z}$.

In fact, $13 \mid (p^2+1)$, for all *p* with $p = 13P + 5, 8$ where $P \in \mathrm{Z}$. Thus it *doesn't follow* that $\sqrt{13}$ *doesn't have* any L–approximations (the real force of this apparently trivial observation later revealed itself in connection with $\sqrt{34}$. How?) The same observation applies to <u>all</u> other cases: $\sqrt{5}, \sqrt{17}, \sqrt{29}, \sqrt{37}, \sqrt{41}, \sqrt{53}, \sqrt{61}, \sqrt{73}, \ldots$ (Fermat, Euler, …)

What about $\sqrt{8}$? Someone made a correct argument by considering the eight possible remainders that ‘*p*’ leaves on division by 8, but that too was improved upon by someone who observed that it was only necessary (as in (iii) with $\sqrt{6}$) to look at the four remainders on division by 4: "if $\frac{p}{q}$ is an L – approx. to $\sqrt{8}$, then $p^2+1=8q^2=4(2q^2)$, impossible since $4\nmid(p^2+1)$."

We now knew that $\sqrt{8},\sqrt{12}$ (already known because of the '3'), $\sqrt{20},\sqrt{24}$ (already known because of the '3'), $\sqrt{28}$ (already known because of '7', but now seen to be more 'economically' argued because of the '4'), $\sqrt{32},\sqrt{40},\sqrt{44}$ ('11' etc.).

(v) I had given then a 'Table of Solutions for Pell's Equation' (see Appendix One: I also told them that Pell's only claim to fame was that Euler had mistakenly attached Pell's name to it. See, for example, the magnificent **[6]**, not just for the Pell reference) and now I asked for explanations for all those entries, starting at $d = 3$, with 'no solution'.

When e.g. I called out '7', I was told "it's because of 7 itself", and when I called out , e.g. '22', I was told "it's because of 11", but when we got down to 34 there was the inevitable blank. It was immediately clear that one couldn't argue with the '17', and certainly not with the '2', and the suggestion – "don't try to argue the 2 and 17 *separately* but instead argue with the 34 itself – was also seen to get us nowhere when we considered:

$p \in \mathrm{Z},\ p = 34P + 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots,\ 31,\ 32,\ 33,$ some $P \in \mathrm{Z}.$

$p^2+1 = 34A + 1,\ 2,\ 5,\ 10,\ 17,\ 26,\ 3,\ 16,\ 31,\ 14,\ 33,\ 20,\ 9,\ \underline{0},\ \ldots$

that '$\underline{0}$' corresponding to $p=(34P+13)$ (and another '0' occurs further along from $p=(34P+21)$). We returned to that later.

(vi) Now began the most important part of the whole day's work. I went back to the *simple numerical fact* that $\frac{3}{2}$ is the first R – approximation to $\sqrt{2}$. Then I wrote on the board with obvious commentary:

$$3^2 - 2.2^2 = 1$$

$$(3-2.\sqrt{2})(3+2.\sqrt{2}) = 1 \text{ 'cross – terms' cancel}$$

$$\underline{(3-2.\sqrt{2})(3+2.\sqrt{2})} = 1 \text{ and multiplication gives:}$$

$$(9-6.\sqrt{2}-6.\sqrt{2}+8)(9+6.\sqrt{2}+6.\sqrt{2}+8)=1, \text{ tidies to:}$$

$$(17-12.\sqrt{2})(17+12.\sqrt{2})=1, \text{ in other words to:}$$

$$17^2 - 2.12^2 = 1.\ !!!!!$$

This *simple* calculation had produced – fluke? chance? accident? miracle? – the *second* R–approximation to $\sqrt{2}$ from the first one! (As the late genius, English comedian, Tommy Cooper , used to say, "just like that…") It's a miracle. Of course to understand it and its developments requires insight and thought, and that is something that will only come to them in time.

And what if we did with $\frac{17}{12}$ what we've just done with $\frac{3}{2}$? Some said we'd get the *third* R–approximation to $\sqrt{2}$. Well, try it and see, and this is what they got:

$$\begin{aligned}
17^2 - 2.12^2 &= 1\\
(17 - 12.\sqrt{2})(17 + 12.\sqrt{2}) &= 1\\
\underline{(17 - 12.\sqrt{2})(17 + 12.\sqrt{2}) = 1}&\\
(289 - 204.\sqrt{2} - 204.\sqrt{2} + 288)(289 + 204.\sqrt{2} + 204.\sqrt{2} + 288) &= 1,\\
(577 - 408.\sqrt{2})(577 + 408.\sqrt{2}) &= 1,\\
577^2 - 2.408^2 &= 1,
\end{aligned}$$

producing, *not* the third R–approximation to $\sqrt{2}$ (which they already knew to be $\frac{99}{70}$), but rather the *fourt*h R – approximation to $\sqrt{2}$, $\frac{577}{408}$.

So where was the third one? Miracle time again!

$$\begin{aligned}
(3^2 - 2.2^2) = 1 \text{ and } (17^2 - 2.12^2) = 1 &\text{ give:}\\
(3 - 2.\sqrt{2})(3 + 2.\sqrt{2}) = 1 &\text{ and}\\
\underline{(17 - 12.\sqrt{2})(17 + 12.\sqrt{2}) = 1}, &\text{ multiplication gives:}\\
(51 - 34.\sqrt{2} - 36.\sqrt{2} + 48)(51 + 34.\sqrt{2} + 36.\sqrt{2} + 48) = 1,&\\
\text{tidies to } (99 - 70.\sqrt{2})(99 + 70.\sqrt{2}) = 1, &\ (\underline{\text{won}} - \underline{\text{der}} - \underline{\text{ful}})!\\
99^2 - 2.70^2 = 1,&
\end{aligned}$$

now producing the third R–approximation to $\sqrt{2}$, $\frac{99}{70}$.

(vii) They took to this immediately. What sort of language was used? I said we were using 'the method of composition' (a made–up term). I didn't want to use the classic identities:

$$(x^2 - 2y^2)(X^2 - 2Y^2) = (xX + 2yY)^2 - 2(xY + yX)^2,$$

which would have been too much out-of-a-hat, but rather wanted to do something so utterly transparent (from a technical point of view), so simple, that almost anyone can follow (though not necessarily appreciate) it.

And what was composition about? It was about: taking an equation of the R–type, *splitting* it up, and then either *combining* it with itself, *producing* another R–type equation, and thus another R–approximation, or: taking an equation of the R–type, *splitting* it up, and combining it with another similarly split R–type equation and in the process thus *producing* another R–type equation.

And what structure was there to all of this? They correctly guessed that 'the m^{th} R–approximation to $\sqrt{2}$ when composed (combined – call it what you will) with the n^{th}

R–approximation to $\sqrt{2}$ gives the $(m+n)^{th}$ R - approx. to $\sqrt{2}$ and so, as I put it, *behaves like* 'ordinary addition'.

(viii) But this sort of thing doesn't just have limited applicability to $\sqrt{2}$; you can do it with $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ e.g. starting with just the *first* R–approximation to $\sqrt{3}$, $\frac{2}{1}$:

$$2^2 - 3.1^2 = 1 \text{ gives } (2 - 1.\sqrt{3})(2 + 1.\sqrt{3}) = 1$$
$$\underline{(2 - 1.\sqrt{3})(2 + 1.\sqrt{3}) = 1}, \text{ giving:}$$
$$(4 - 2.\sqrt{3} - 2.\sqrt{3} + 3)(4 + 2.\sqrt{3} + 2.\sqrt{3} + 3) = 1, \text{ tidies to:}$$
$$(7 - 4.\sqrt{3})(7 + 4.\sqrt{3}) = 1,\ 7^2 - 3.4^2 = 1,$$

and $\frac{7}{4}$ is the *second* R – approximation to $\sqrt{3}\ldots$ Well it's clear where that leads to…

Of course none of this proves anything of a general nature. I just wanted to expose them to certain phenomena/ideas, to excite their interest and curiosity, not to prove anything.

If you compose the fourth and sixth R–approximations to $\sqrt{2}$ you get the tenth R–approximation to $\sqrt{2}$. The same happens if you compose $\sqrt{2}$'s third and seventh R–approximations. And the same holds if you compose *all* possible R–approximations for all possible irrational square-roots. Why is this true? Later I tried to convey – with a little bit of hand-waving – why this is true.

What about L–approximations? How do they compose? Simple! Let's start at the very beginning. Just take the first L–approximation to $\sqrt{2}$, $\frac{1}{1}$, and then from:

$$(1^2 - 2.1^2) = -1, \text{ obtain}$$
$$\text{splitting } (1 - 1.\sqrt{2})(1 - 1.\sqrt{2}) = -1,$$
$$\underline{(1 - 1.\sqrt{2})(1 - 1.\sqrt{2}) = -1}$$
$$\text{combining } (1 - 1.\sqrt{2} - 1.\sqrt{2} + 2)(1 + 1.\sqrt{2} + 1.\sqrt{2} + 2) = 1,$$
$$(3 - 2.\sqrt{2})(3 + 2.\sqrt{2}) = 1,$$
$$3^2 - 2.2^2 = 1. \text{ !!!!!!!}$$

It was immediately clear that something was going on there.

And if one had chosen a different L–approximation to $\sqrt{2}$ and composed it with itself in like manner? The <u>second</u> L–approximation to $\sqrt{2}$, $\frac{7}{5}$, produces what? It's clear in advance it will produce an R–approximation to $\sqrt{2}$ (as the –1 times –1 gives +1), but which one? One might suspect it will give the second R–approximation to $\sqrt{2}$, but actual calculation shows it produces the R–approximation $\frac{99}{70}$, which is the *third* one. So where has the second R–approximation gone to? Ah! yes!, you get it by composing/combining the first and second L–approximations to $\sqrt{2}$:

$$(1^2 - 2.1^2) = -1 \text{ and } (7^2 - 2.5^2) = -1 \text{ and}$$
$$(1 - 1.\sqrt{2})(1 + 1.\sqrt{2}) = -1$$
$$\underline{(7 - 5.\sqrt{2})(7 + 5.\sqrt{2}) = -1},$$
$$(7 - 7.\sqrt{2} - 5.\sqrt{2} + 10)(7 + 7.\sqrt{2} + 5.\sqrt{2} + 10) = 1,$$
$$(17 - 12.\sqrt{2})(17 + 12.\sqrt{2}) = 1, \text{ etc.}$$

And what if you compose an L– and an R–approximation to $\sqrt{2}$? It's obvious that you get an L–approximation. And the overall structure jumps out at you. And many quickly grasped the point that just as the composition of R–approximations behaves like addition with 1, 2, 3, 4, … so too does the joint composition of L's and R–'s behave like addition, but now with $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, etc.

And what of $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ Well it was immediately clear that from, e.g. the first R–approximation $\frac{2}{1}$ to $\sqrt{3}$, you could do just this:

$$(2^2 - 3.1^2) = 1, \ (2 - 1.\sqrt{3})(2 + 1.\sqrt{3}) = 1, \text{ which led to:}$$
$$(4 - 2.\sqrt{3} - 2.\sqrt{3} + 3)(4 + 2.\sqrt{3} + 2.\sqrt{3} + 3) = 1, \text{ tidying up}$$
$$\text{to } (7 - 4.\sqrt{3})(7 + 4.\sqrt{3}) = 1, \ 7^2 - 3.4^2 = 1.$$

But, *more generally*, if $\frac{p}{q}$ is an R – approximation to $\sqrt{3}$, then from $(2^2 - 3.1^2) = 1$ and $(p^2 - 3.q^2) = 1$ you get:

$$(2 - 1.\sqrt{3})(2 + 1.\sqrt{3}) = 1, \text{ and}$$
$$\underline{(p - q.\sqrt{3})(p + q.\sqrt{3}) = 1},$$
$$(2p - p.\sqrt{3} - 2q.\sqrt{3} + 3q)(2p + p.\sqrt{3} + 2q.\sqrt{3} + 3q) = 1,$$
$$[(2p + 3q) - (p + 2q).\sqrt{3}][(2p + 3q) + (p + 2q).\sqrt{3}] = 1,$$
$$(2p + 3q)^2 - 3(p + 2q)^2 = 1,$$

and thus producing the R–approximation $\frac{2p+3q}{p+2q}$ to $\sqrt{3}$. Since this was the law that had been discovered empirically away back in our first week, the reader can imagine the great sense of satisfaction there was amongst the sensitive ones. (Also it will be seen how this relates to my earlier introductory remarks concerning the problem of trying to find R–approximations to $\sqrt{19}$).

All the obvious correct general guesses were made, but I didn't wish our work to only be at this level of blind calculation and guessing: I wished – and it had to be done quickly as our time was running out – to convey *some* idea as to why, for example, the repeated composition/combining of the 'fundamental solution' of the Fermat–Pell equation produces all of the R–approximations. To <u>convey</u> the idea I took the first and third R–approximations to $\sqrt{2}, \frac{3}{2}$ and $\frac{99}{70}$, but *instead* of combining them by:

$$(3-2.\sqrt{2})(3+2.\sqrt{2})=1$$
$$\underline{(99-70.\sqrt{2})(99+70.\sqrt{2})=1}, \text{ to give}$$
$$(297-198.\sqrt{2}-210.\sqrt{2}+280)(297+198.\sqrt{2}+210.\sqrt{2}+280)=1,$$
$$(577-408.\sqrt{2})(577+408.\sqrt{2})=1,\ (577^2-2.408^2)=1,$$

which – as they expected – produces the *fourth* R–approximation to $\sqrt{2}$, *now* combine them like this:

$$(3+2.\sqrt{2})(3-2.\sqrt{2})=1, \text{ sign interchange!}$$
$$\underline{(99-70.\sqrt{2})(99+70.\sqrt{2})=1}, \text{ (no change) gives}$$
$$(297+198.\sqrt{2}-210.\sqrt{2}-280)(297-198.\sqrt{2}+210.\sqrt{2}-280)=1,$$
$$\text{tidies to } (17-12.\sqrt{2})(17+12.\sqrt{2})=1,\ 17^2-2.12^2=1,$$

and so only produces the *second* R–approximation to $\sqrt{2}$.
I used very informal language, saying that we had 'pulled back a solution'. But this pulling back idea would go through not just with $\sqrt{2}$, but can be seen to go through generally with $\sqrt{3},\sqrt{5},\sqrt{6},\ldots$ And it was this that enabled one to prove not only that the <u>only</u> R–approximations to $\sqrt{2}$ are those generated by *repeated* composition (but without the switching of signs) of the 'fundamental solution' (i.e. the numerator and denominator of the first R–approximation of the Fermat–Pell equation with itself). This can be seen to be so because <u>if</u> there were any R–approximations other than the ones so generated, *then*, thinking about the one of those with the least denominator, and composing it with the fundamental solution – *with sign interchange* – produces an R–approximation (that bit is obvious) which has a smaller denominator (that's a detail to be argued) than the one under consideration, but which can't be one of the R–approximations generated by repeated composition of the fundamental solution with itself (easily argued). Producing this smaller denominator provides a refutation…

(ix) Finally, since many of them seemed to be so taken with the idea of continued fractions, I gave them a copy of a table of continued fraction expansions (Appendix Two) from **[7]** (later I was asked to recommend a book on Number Theory, and I was tempted to name the classic **[8]** [where I had myself first feasted], but felt that it might prove to be a bit much for them, and it was the Davenport I suggested, partly because it is such a wonderful book, but also because Davenport was the first great number–theorist that I ever set eyes on…).

I explained the *meaning* of the table. Those '–1's' and '+1's' related exactly to there being (for '–1') L–approximations and (for '+1':) there being no L–approximations.

Those (x, y) = (1, 1), (2, 1), (2, 1), (5, 2), opposite the values 2, 3, 4, 5, 6, etc. were just the fundamental solutions of the obvious Fermat–Pell equations…

That '1;2' as the 'continued fraction for '$\sqrt{2}$' *meant*:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$ the '2' get's repeated

etc. with convergents:

$$1,\quad 1+\frac{1}{2},\quad 1+\cfrac{1}{2+\cfrac{1}{2}},\quad 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}},\quad \ldots\ldots, \text{ namely,}$$

$$\tfrac{1}{1},\quad \tfrac{3}{2},\qquad \tfrac{7}{5},\qquad \tfrac{17}{12},\qquad \ldots\ldots$$

the L–and R–approximations to $\sqrt{2}$!!!!!

Did that always happen? No! If you calculate with the continued fraction for $\sqrt{3}$, namely '1;1,2', which *means*:

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}}$$ (the '1, 2' gets repeated)

etc. with convergents

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \ldots$$

Of course these couldn't possible be L– and R–approximations to $\sqrt{3}$ since $\sqrt{3}$ hasn't got any L–approximations, but these convergents *do* give rational approximations to $\sqrt{3}$ which alternate to the left and right of $\sqrt{3}$. That, *e.g.*, $\frac{19}{11}$, which is less than $\sqrt{3}$, instead of being a solution of the equation $p^2 = 3q^2 - 1$, *is* a solution of the next best one, namely $p^2 = 3q^2 - 2$ …! And, every other convergent starting with $\frac{1}{1}$ does the same!…

Also, every other convergent starting with $\frac{2}{1}$, is an R – approximation to $\sqrt{3}$…

So, in their own time there were many, many things to investigate … .

Did they notice anything just by *looking* at those continued fractions? Of course. It would be impossible not to see. It was noticed that every one of them had its repeated part (the part after the ';') end in a number which is double the initial part (the part before the ';'). It was also noticed that when you "cover over the first and last parts, the other bit is symmetrical" So, *e.g.*, for $\sqrt{31}$ the "other bit is (1, 1, 3, 5, 3, 1, 1) and for $\sqrt{29}$ it is (2, 1, 1, 2).

I asked if they noticed that sometimes the other bit had a central *isolated* terms (5 in the case of $\sqrt{31}$) and somtimes it didn't (as in the case of $\sqrt{29}$). Could they see anything special about these occasions? Of course! The first happens when there are no L–approximations and the second when there are! I mentioned Euler and Lagrange…

And what about $\sqrt{34}$, of which we had earlier seen that we couldn't argue that it has no L–approximations by a simple approach. Of course if we took for granted unproved results about continued fractions we could get a proof (the continued fraction expansion of $\sqrt{34}$ has an isolated central term), but could we get it any other way? Richard pointed out that it we took for granted the earlier discovered (but unproved) result that the fundamental L–approximation (<u>if</u> there were any L–approximations) when composed with itself gives the fundamental R–approximation, then in the case of $\sqrt{34}$ we could argue that it has no L–approximations by *supposing* that it did, and by noting that when composed with itself it would come to $\frac{35}{6}$ (see table) it would have to have denominator less than 6, and so would be either 1, 2, 3, 4 or 5, and by just checking each of these in turn and seeing that none of them give rise of an L–approximation then there are none…

I also just mentioned that every number has a 'continued fraction expansion', that rational numbers have terminating expansions, and irrational numbers have non–terminating ones. Finite and infinite expansions if you will allow. But they should note that I *hadn't* shown them <u>how</u> these expansions are arrived at.

Now, if only I had asked Alaistair Wood to allow me just one more day…

Thinking back to that time (it is now November 4th 1993 and I have been typing this report on and off since the end of July) I can scarcely believe that most of them took in as much as they did, and my admiration of them – especially the younger ones – knows no bounds.

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I end with two tables

1. Some solutions of *Fermat–Pell* equations

2. Some standard continued fraction expansions

d	ε = least solution of $x^2 - dy^2 = -1$	δ = least solution of $x^2 - dy^2 = 1$
2	(1, 1)	(3, 2)
3	No solution	(2, 1)
5	(2, 1)	(9, 4)
6	No solution	(5, 2)
7	No solution	(8, 3)
8	No solution	(3, 1)
10	(3, 1)	(19, 6)
11	No solution	(10, 3)
12	No solution	(7, 2)
13	(18, 5)	Find
14	No solution	(15, 4)
15	No solution	(4, 1)
17	(4, 1)	Find yourself
18	No solution	(17, 4)
19	No solution	(170, 39)
20	No solution	(9, 2)
21	No solution	(55, 12)
22	No solution	(197, 42)
23	No solution	(24, 5)
24	No solution	(5, 1)
26	(5, 1)	Find
27	No solution	(26, 5)
28	No solution	(127, 24)
29	(70, 13)	Find
30	No solution	(11, 2)
31	No solution	(1520, 273)
32	No solution	(17, 3)
33	No solution	(23, 4)
34	No solution	(35, 6)

35	No solution	(6, 1)
37	(6, 1)	Find
38	No solution	(37, 6)
39	No solution	(25, 4)
40	No solution	(19, 3)
43	No solution	(3482, 531)
46	No solution	(24335, 3588)
53	There is a solution	(find yourself, 9100)
61	There is a solution	(find yourself, 226153480)
62	No solution	(find yourself, 8)
109	There is a solution	(find yourself, 15140424455100)
110	No solution	(21, 2)
421	There is a solution	Here *y* has 33 digits!!!!
433	There is a solution	Here *y* has 19 digits.

N	***Continued fraction for*** $\sqrt{N}$	***x***	***y***	$x^2 - Ny^2$
2	1;2	1	1	–1
3	1;1,2	2	1	+1
5	2;4	2	1	–1
6	2;2,4	5	2	+1
7	2;1,1,1,4	8	3	+1
8	2;1,4	3	1	+1
10	3;6	3	1	–1
11	3;3,6	10	3	+1
12	3;2,6	7	2	+1
13	3;1,1,1,1,6	18	5	–1
14	3;1,2,1,6	15	4	+1
15	3;1,6	4	1	+1
17	4;8	4	1	–1
18	4;4,8	17	4	+1
19	4;2,1,3,1,2,8	170	39	+1
20	4;2,8	9	2	+1
21	4;1,1,2,1,1,8	55	12	+1
22	4;1,2,4,2,1,8	197	42	+1
23	4;1,3,1,8	24	5	+1
24	4;1,8	5	1	+1
26	5;10	5	1	–1
27	5;5,10	26	5	+1
28	5;3,2,3,10	127	24	+1
29	5;2,1,1,2,10	70	13	–1
30	5;2,10	11	2	+1
31	5;1,1,3,5,3,1,1,10	1520	273	+1
32	5;1,1,1,10	17	3	+1
33	5;1,2,1,10	23	4	+1
34	5;1,4,1,10	35	6	+1

35	5;1,10	6	1	+1
37	6;12	6	1	–1
38	6;6,12	37	6	+1
39	6;4,12	25	4	+1
40	6,3,12	19	3	+1
41	6;2,2,12	32	5	–1
42	6;2,12	13	2	+1
43	6;1,1,3,1,5,1,3,1,1,12	3482	531	+1
44	6;1,1,1,2,1,1,1,12	199	30	+1
45	6;1,2,2,2,1,12	161	24	+1
46	6; 1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12	24335	3588	+1
47	6; 1, 5, 1, 12	48	7	+1
48	6; 1, 12	7	1	+1
50	7; 14	7	1	–1