Ramsey numbers

Ramsey numbers (formally defined shortly) are the outcome of work of the brilliant English mathematician-philosopher-economist Frank Ramsey[1]. In a seminal paper[2] he proved the following result:

> Let G be an infinite simple[3] graph, then G has an infinite sub-graph G' **every two** of whose vertices are joined[4], or an infinite sub-graph G' **no two** of whose vertices are joined[5]

Erdös[6] and Szekeres formulated a finite version of Ramsey's result, known as the Erdös-Szekeres theorem. In crude simplistic terms, their theorem asserts that every finite simple graph, having at *least* a *certain* number of vertices, must contain either a clique of a *certain* size or an independent set of a *certain* size. More precisely – and only by way of an introductory example – their theorem asserts that if one chooses two numbers 7 and 8 (say), then every simple graph G having at *least* a *certain* number of vertices (N, say) **must** contain **either** a clique of size 7, **or** an independent set of size 8. The *smallest* such 'N' for which that is true is known as the Ramsey number $r(7, 8)$. Remarkably, $r(7, 8)$ – and all other Ramsey numbers *like* it – **does exist** (as you will shortly see), but nobody[7] knows its *exact* value!!

Most people encounter Ramsey number (perhaps without knowing so) through the well-known **six-people-at-a-party-problem**: prove that for any six people there must be at least three of them, every two of whom know each other, or three of them, no two of whom know each other (and – as a pointer to the *significance* of the '6' in relation to the '3' and '3' – construct an example of '5' people with no 3-clique and no 3-independent set. Care should be taken if introducing this problem to non-mathematicians, or, more precisely, acquaintances lacking the facility for abstract thought.)

In these notes I am summarizing work discussed at length in class and am resorting to graph theory language, with the obvious correspondences being understood:

knowing $\leftrightarrow$ joined
not knowing $\leftrightarrow$ not joined

(You ought to realise that resorting to such language is – for you and I – merely a convenience, and is not an integral part of what we are thinking about. To put it another way, should you ever try to introduce someone to Ramsey numbers, don't start by rambling on about 'graphs', 'vertices', 'edges', 'cliques', 'independent sets', unless you wish to frighten everyone off.)

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[1] http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Ramsey.html
[2] On a Problem of Formal Logic, *Proceeding of the London Mathematical Society*, **30** (1930), 264-286.
[3] A graph is *simple* if every two of its vertices are either joined by a *single* edge, or are not joined at all.
[4] Forming what is knows as an infinite *clique*.
[5] Forming what is knows as an infinite *independent set*.
[6] http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Erdos.html
[7] Recall Erdös – talking about Ramsey numbers in the video *N is a Number* – remarking (in wonderment) that not even the value of $r(5, 5)$ is known, though it is known to lie between 43 and 49. $r(7, 8)$ is known to lie between 216 and 1031; quite a range!!

Solution (expressed in graph theory language) to the six-people-at-a-party-problem. Let G be a simple graph with 6 vertices, and let P be any one of those vertices. Since P is joined to 5 or 4 or 3 or 2 or 1or 0 of the other 5 vertices then

1. P **is** joined to at least 3 of the other vertices, or
2. P is **not** joined to at least 3 of the other vertices

In case (1), if **some** 2 of the other vertices (P' and P'' say) **are** joined, then **every** pair from P, P' and P'' are joined: they form a 3-clique (the people they represent are 'mutually acquainted'); otherwise **no** 2 of the other 3 vertices (P', P'', P''' say) are joined: they form a 3-independent set (the people they represent are 'mutually **not** acquainted'.

In case (2), if **some** 2 of the other vertices (P' and P'' say) are **not** joined then **no** pair from P, P' and P'' are joined: they form a 3-independent set (the people they represent are 'mutually **not** acquainted'); otherwise **every** 2 of the other 3 vertices (P', P'', P''' say) **are** joined: they form a 3-clique (the people they represent **are** 'mutually acquainted').

Definition. Let $k, l \geq 2$, and suppose there is a number N such that every simple graph with N vertices has either a clique of size k or an independent set of size l, then the minimum such N is called[8] the *Ramsey number* $r(k, l)$.

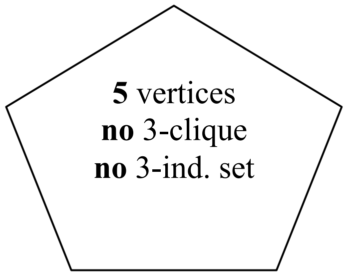
Examples.

1. A trivial one: $r(k, 2) = k$ for all $k \geq 2$. **Why**? It's immediate: let G be **any** simple graph having **exactly** k vertices. If **all** G's vertices are mutually joined then G automatically has a k-clique (itself!), while if **not all** of G's vertices are mutually joined then **some** 2 of G's vertices are not joined, and so G automatically has a 2-independent set.
   It is **obvious** that one may construct a simple graph with *fewer* than k vertices, with no k-clique, and no 2-independent set. Thus $r(k, 2) = k$.

2. A trivial one: $r(2, l) = l$ for all $l \geq 2$. **Why**? It's immediate: let G be **any** simple graph having **exactly** l vertices. If **some 2** of G's vertices are mutually joined then G automatically has a 2-clique, while if **no 2** of G's vertices are joined then G automatically has an l-independent set (itself!)
   It is **obvious** that one may construct a simple graph with *fewer* than l vertices, with no 2-clique, and no l-independent set. Thus $r(2, l) = l$.

3. $r(3, 3) = 6$. **Why**? Well we already know that $r(3, 3)$ is at most 6, and all we have to do to show that it **is actually** 6 is to make up an example of a simple graph, having 5 vertices, which has **no** 3-clique and **no** 3-independent set. The obvious example[9] is provided by the **simple pentagon**:

[8] Actually I have only introduced what might be called the **2-variable** Ramsey numbers. There are several-variables generalizations (in which **multi-colouring** plays a part).
[9] Such a graph is called a *Ramsey graph*. Not surprisingly they have lovely symmetry.



4. It should be **obvious** that $r(k, l) = r(l, k)$. One should **see** that it is true. 'Knowing' and 'not-knowing', being 'joined' and 'not being joined', being joined by 'red' (say, to represent 'knowing') or 'blue' (say, to represent 'not knowing'), … , are, abstractly, the **same**.

The Erdös-Szekeres theorem. Suppose $r(k-1, l)$ and $r(k, l-1)$ both exist $(k, l \geq 2)$, then $r(k, l)$ also exists and $r(k, l) \leq r(k-1, l) + r(k, l-1)$.

Proof. Let *G* be a simple graph with $r(k-1, l) + r(k, l-1)$ vertices[10], and let *P* be any one of those vertices. Then, either

(*a*) *P* **is** joined to at least $r(k-1, l)$ of the other vertices[11]. Thus **either some** $(k-1)$ of the $r(k-1, l)$ vertices form a $(k-1)$-clique – in which case *P*, together with those $(k-1)$ vertices, form a *k*-clique – **or some** *l* of those $r(k-1, l)$ vertices form an *l*-independent set. In either event *G* has a *k*-clique or an *l*-independent set.

or

(*b*) *P* **is not** joined to at least $r(k, l-1)$ of the other vertices[12]. Then **either some** *k* of the $r(k, l-1)$ vertices form a *k*-clique, **or some** $(l-1)$ of those $r(k, l-1)$ vertices form an $(l-1)$-independent set, in which case *P*, together with those $(l-1)$ vertices, form an *l*-independent set. In either event *G* has a *k*-clique or an *l*-independent set.

That completes the proof.

Comment. One should recall from our discussions how **absolutely critical** it is to assert the **correct minima** in (*a*) and (*b*) (the 'lift' that I refer to in footnotes 11 and 12 should be your guiding principle). For consider the first non-trivial examples after $r(3, 3)$, either $r(4, 3)$ or $r(3, 4)$. If one **tries** to argue (e.g.) that

$$r(4, 3) \leq r(4-1, 3) + r(4, 3-1)$$

namely

$$r(4, 3) \leq 10$$

[10] In the $r = 3$ and $l = 3$ case the '6' was the sum of $r(2, 3) = 3$ and $r(3, 2) = 3$.

[11] The key idea now is to give a 'lift' – as it were – to that $(k-1)$.

[12] The key idea now is to give a 'lift' to that $(l-1)$.

then one **could** *attempt* to proceed by saying of a vertex P (in a 10 vertex simple graph) that it must be joined to 9 or 8 or … or 2 or 1 or 0 of the other 9 vertices. And one **could** then (correctly) assert that P must be joined to a minimum of 5 vertices, or not joined to a minimum of 5 vertices. However, to do so, would lead one **nowhere** (in terms of arguing to the desired conclusion that $r(4, 3) \leq 10$).

Even if one were to (correctly) assert that P must be joined to a minimum of **4** vertices, or not joined to a minimum of **6** vertices, that **too** would lead one **nowhere** (in terms of arguing to the desired conclusion that $r(4, 3) \leq 10$). Of course it **would** enable one to argue that $r(3, 4) \leq 10$ (note the 'switch' of the '4' and '3'). This is an important point to absorb, in terms of one's personal understanding, and you should recall the (almost interminable!) struggle over that very point in class discussions.

Note. If $r(k-1, l)$ and $r(k, l-1)$ are **both** even, it can be argued that a little more is true, namely, $r(k, l) < r(k-1, l) + r(k, l-1)$. Thus it follows (e.g.) that $r(4, 3) \leq 9$, and – in fact – $r(4, 3) = 9$, as is shown by exhibiting a 'Ramsey graph' having 8 vertices (namely one with no 4-clique and no 3-independent set).

Which Ramsey numbers are *known*[13]? Very few non-trivial (meaning, of course, that $k, l \geq 3$) Ramsey numbers are known, despite huge efforts at determining them.

However one can at least say something (using the Erdös-Szekeres theorem) about **how big** they are, *at most*. For example one may easily (though crudely) argue that $r(5, 5)$ is at most 70. How? Simply by making a succession of applications of the E-S theorem:

- $r(4, 3) \leq r(3, 3) + r(4, 2) = 6 + 4 = 4.$
  $\therefore\ r(4, 3) \leq 10.$
- $r(4, 4) \leq r(3, 4) + r(4, 3) \leq 10 + 10 = 20.$
  $\therefore\ r(4, 4) \leq 20.$
- $r(5, 4) \leq r(4, 4) + r(5, 3)$ [see separate calculation] $\leq 20 + 15 = 35.$
  $[r(5, 3) \leq r(4, 3) + r(5, 2) \leq 10 + 5 = 15]$
  $\therefore\ r(5, 4) \leq 35.$
- $r(5, 5) \leq r(4, 5) + r(5, 4) \leq 35 + 35 = 70.$
  $\therefore\ r(5, 5) \leq 70.$

Comment. Of course one may easily improve the last inequality by using actual **known** results for $r(4, 3)$, $r(4, 4)$ and $r(5, 3)$: $r(4, 3) = 9$, $r(4, 4) = 18$, $r(5, 3) = 14$. In fact, it happens to be known that $43 \leq r(5, 5) \leq 49$.

For the record. At the time of writing, here are the only Ramsey numbers (the 2-variable ones, that is, and with $k, l \geq 3$) whose values are known:

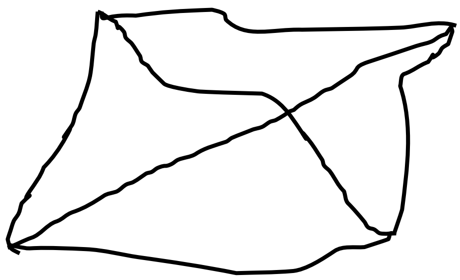
[13] A good web reference is http://mathworld.wolfram.com/RamseyNumber.html

$r(3,3)$	6
$r(3,4)$	9
$r(3,5)$	14
$r(3,6)$	18
$r(3,7)$	23
$r(3,8)$	28
$r(3,9)$	36
$r(4,4)$	18
$r(4,5)$	25

Of the more general (non-trivial) Ramsey numbers, **only one** is known!! It is $r(3,3,3)$, whose value is known to be 17. The **meaning** of that is that if one chooses any 17 (or more) points and joins every two of them using any one of **3** colours, red, green, and blue (say; by the way, that '3'(colours) is the **number** of variables – coordinates – and **not** the '3' in that '3, 3, 3'), then, however one does it, there will always result either a red triangle, a green triangle, or a blue triangle (that's what the '3, 3, 3' is about). Recall that an **interpretation** of $r(3,3)$ being 6 is that if one chooses any 6 (or more) points, and joins every two of them using **2** colours, red and blue (say), then, however one does it, there will always result **either** a red triangle **or** a blue triangle.

You should recall, too, how that may returned into a game (which you could play with your friends, introduce to children, use your imagination…) for 2 people who have paper and two differently coloured crayons. On the paper mark 6 points, and then **play**. A 'move' is to join two points by an edge (it doesn't have to be 'straight'). Players move alternately, using their colour. Who 'loses'? The first one to complete a 'triangle' (a 3-clique, call it what you will). By Ramsey theory there **must** be a loser[14].

It would, of course, be more interesting to play with 18 points!! There, the loser is the one who first completes a 4-clique:



Once again, by Ramsey theory, there **must** be a loser.

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[14] Must the player who moves first be the loser, providing the second player plays appropriately?