COLÁISTE PHÁDRAIG, BAILE ÁTHA CLIATH
(Coláiste de chuid Ollscoil Chathair Bhaile Átha Cliath)

ST PATRICK'S COLLEGE, DUBLIN
(A College of Dublin City University)

SUMMER EXAMINATIONS 2005

B.A. Degree in Humanities Examinations

Mathematics (Challenging Mathematical Puzzles and Problems)

Fourth Paper

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Time: 3 hours

Answer **five** questions

1 (a) Boxes numbered 1 to n are all locked, and are visited by n people. The r^{th} person $(1 \le r \le n)$ first visits box number numbered r and subsequently every r^{th} box. If the visited box is already locked, they proceed to unlock it, and vice-versa. Which boxes are unlocked after all n people have finished their business?

(b) Prove that every prime (except 2 and 5) divides infinitely many terms of the (decimal) sequence: 11, 111, 1111, 11111, …

(c) (i) Verify that each of the first four terms of the sequence $11_3, 111_3, 1111_3, \ldots,$ is a sum of two squares of integers.

(ii) Prove that no integer $n \equiv 3 \pmod 4$ is a sum of two squares, and deduce that no term of the (decimal) sequence 11, 111, 1111, 11111, is a sum of two squares of integers.

2 (a) Let $a, b \in \mathbf{N}$ with $a < b$, $a \nmid b$, and let $\frac{1}{n}$ be the nearest unit-fraction to $\frac{a}{b}$ from below. Explain what that means, and prove that the numerator of the fraction $\frac{an-b}{bn}$ is smaller than that of $\frac{a}{b}$.

(b) Show that the egyptian greedy algorithm applied to $\frac{4}{25}$ results in maximum possible numerators during successive stages of its implementation.

(c) Apply the (unproven in general) egyptian *odd* greedy algorithm to $\frac{2}{7}$.

(d) For integral $n > 1$, prove that $\sum_{r=2}^{n} \frac{1}{r}$ is not an integer.

3 (a) Briefly explain the game **Nim**, and analyse the $\{1, 2, 3\}$ position.

(b) Introduce Bouton's concept of *correct* and *incorrect* positions in the game N**im**, and prove that any move from a correct position leads to an incorrect position.

(c) How many winning moves are there from the incorrect **Nim** position $\{30, 25, 13\}$? Find winning moves for each of them.

4 (a) Prove that out of any six people there must exist three, every two of whom are acquainted, or three, no two of whom know each other.

(b) Define the Ramsay numbers $r(k, l)$ $(k, l \geq 2)$ and prove the Erdös-Szekeres theorem that if $r(k-1, l)$ and $r(k, l-1)$ both exist, then $r(k, l)$ also exists, and $r(k, l) \leq r(k-1, l) + r(k, l-1)$.

(c) Show that $r(3, 3) = 6$.

(d) Given that strict inequality holds in (b) whenever $r(k-1, l)$ and $r(k, l-1)$ are both even, prove that $r(3, 4) = 9$.

5 (a) Use elementary Linear Algebra methods to determine a quadratic $q(x)$ such that $q(0) = 2$, $q(1) = -1$, and $q(2) = 4$.

(b) If $P(x)$ denotes a polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, n$, determine $P(n+1)$. [USA Mathematical Olympiad, 1975]

(c) Find a polynomial $P(x)$ of degree n such that $P(k) = \frac{k+1}{k}$ for $k = 1, 2, \ldots, n+1$, and determine $P(n+2)$.

6 (a) Let a and b be distinct; find B and C such that $P(x) = x^2 + Bx + C$ satisfies $P(a) = b$ and $P(b) = a$.

(b) Let a, b, and c denote three distinct integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. [USA Mathematical Olympiad, 1974]

(c) State and prove an n distinct integer generalisation of the result in (b).

7 (a) You are the assistant in a performance of Eppstein's variation of the W. F. Cheney card trick (using the standard 52-card pack), and a challenger C has passed you the following *four* cards: 10♥, J♦, 7♠, 8♣. What signal can you give with just three of those four cards that would enable your accomplice-performer P to identify the other card? Explain how P would identify the

other

card. (Use C, D, H and S for ♣, ♦, ♥ and ♠, and you may assume in your explanation that a reader is already familiar with the details of the original Cheney solution.)

(b) State the n from $(n! + n - 1)$ improved generalization of the Cheney card trick.

In the $n = 5$ case of the $(n! + n - 1)$ generalization, suppose C chooses 11, 19, 46, 98 and 113 from the range 0 to 123, and passes them to you, the assistant. Which four of those numbers, and in what order, would you pass to your accomplice-performer, P, that would enable P to identify the number you still hold? Show all relevant calculations.