**COLÁISTE PHÁDRAIG, BAILE ÁTHA CLIATH**
**(Coláiste de chuid Ollscoil Chathair Bhaile Átha Cliath)**

ST PATRICK'S COLLEGE, DUBLIN
(A College of Dublin City University)

SUMMER EXAMINATIONS 2005

Second University Examination for the B.A. Degree

Mathematics (Challenging Mathematical Puzzles and Problems)

Fourth Paper

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Time: 3 hours

Answer **five** questions

1 (a) An $8 \times 8$ grid of unit squares may be covered by 32 dominoes. Prove, however, that if two opposite corners of the grid are removed, the remaining 62 squares cannot be covered by 31 dominoes.

(b) Suppose that each square of a $4 \times 7$ chessboard, as shown below, is coloured either black or white. Prove that with *any* such colouring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board such as the one outlined in the figure) whose four distinct corner squares are all of the same colour:

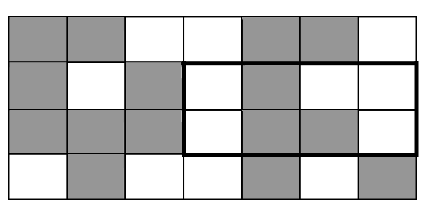

Exhibit a black-white colouring of a $4 \times 6$ board in which the four corners of every rectangle, as described above, are not all of the same colour. [USA Mathematical Olympiad, 1976]

2 (a) Complete any three entries in the following *Sodoku* puzzle, giving a justification for each of your entries:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | 3 |
| | 7 | | 5 | 1 | 6 | | 8 | |
| | 8 | | | | | | 6 | |
| | | 5 | 4 | | 2 | 9 | | |
| | 3 | | | | | | 4 | |
| | | 7 | 3 | | 9 | 6 | | |
| | 5 | | | | | | 9 | |
| | 4 | | 8 | 6 | 5 | | 7 | |
| 6 | | | | | | | | 8 |

(b) Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his neighbours receive jointly. Is there just one way to distribute the money? Prove your answer. [Stanford Mathematics Problems, 1956]

(c) During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, some three were sleeping simultaneously. [USA Mathematical Olympiad, 1986]

3 (a) Let $a, b \in \mathbf{N}$ with $a < b$, $a \nmid b$, and let $\frac{1}{n}$ be the nearest unit-fraction to $\frac{a}{b}$ from below. Explain what that means, and prove that the numerator of the fraction $\frac{an-b}{bn}$ is smaller than that of $\frac{a}{b}$.

(b) Show that the egyptian greedy algorithm applied to $\frac{4}{25}$ results in maximum possible numerators during successive stages of its implementation.

(c) Apply the (unproven in general) egyptian *odd* greedy algorithm to $\frac{2}{7}$.

(d) Prove that no integer can be expressed as a sum of reciprocals of distinct primes.

4 (a) Briefly explain the game **Nim**, and analyse the $\{1, 2, 3\}$ position.

(b) Introduce Bouton's concept of *correct* and *incorrect* positions in the game N**im**, and prove that any move from a correct position leads to an incorrect position.

(c) How many winning moves are there from the incorrect **Nim** position $\{30, 25, 13\}$? Find winning moves for each of them.

5 (a) Let $f(x) = ax^2 + bx + c$ be a polynomial with real coefficients having distinct real roots $\alpha$ and $\beta$. Show that $f'(\alpha) = -f'(\beta)$.

(b) Express $\frac{19x+1}{(2x-1)(3x+2)}$ in the form $\frac{A}{2x-1} + \frac{B}{3x+2}$, for some constants $A$ and $B$.

(c) Let $f(x) = ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients having distinct real roots $\alpha$, $\beta$ and $\gamma$. Show that $\frac{1}{f'(\alpha)} + \frac{1}{f'(\beta)} + \frac{1}{f'(\gamma)} = 0$.

(d) Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial with real coefficients having $n$ distinct real roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Express $f'(\alpha_1)$ in terms of $a_0, \alpha_1, \ldots, \alpha_n$.

6 (a) Find a quadratic $q(x)$ such that $q(0) = 2$, $q(1) = -1$, and $q(2) = 4$.

(b) If $P(x)$ denotes a polynomial of degree $n$ such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, n$, determine $P(n+1)$. [USA Mathematical Olympiad, 1975]

(c) Find a polynomial $P(x)$ of degree $n$ such that $P(k) = \frac{k+1}{k}$ for $k = 1, 2, \ldots, n+1$, and determine $P(n+2)$.

7 (a) Let $a$ and $b$ be distinct; show that if $P(x) = Ax + B$ satisfies $P(a) = b$ and $P(b) = a$, then $A = -1$, and $B = a + b$.

(b) Let $a$, $b$, and $c$ denote distinct integers, and let $P$ denote a quadratic having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. [modified USA Mathematical Olympiad, 1974]

(c) Let $a$, $b$, and $c$ denote three distinct integers, and let $P$ denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. [USA Mathematical Olympiad, 1974]